Precision of Derived Velocity from Kinematic GPS for Land Yachting Speed Record Attempt

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08-May-2022

Abstract

The international governing body for the sport of land and sand yachting was formed in 1962. The Federation International de Sand et Land Yachting (FISLY) has member countries around the world and administers land yachting speed records. The current record is held by Richard Jenkins (UK) in *Greenbird* at a speed of 202.9 km/h set on 26-Mar-2009 at Ivanpah Dry Lake, Prim, Nevada, USA. An attempt to better this record will be made by *Emirates Team New Zealand* in mid-2022 and it is proposed to use Post-Processed Kinematic (PPK) GPS[1] measurements to determine the velocity of the land yacht and this paper will provide an analysis of three methods of determining velocity or speed of the land yacht: (i) from time and coordinate differences obtained from successive PPK measurements and using the simple relationship that velocity is distance divided by time; (ii) from the land yacht's cumulative distance travelled s_k derived from PPK measurements at times t_k for $k = 1,2,3,\ldots$ and a first-order central difference approximation of velocity; and (iii) from a Kalman Filter using local plane coordinates E_k, N_k derived from PPK measurements at times t_k.

Introduction

The current speed record for a land yacht is 202.9 km/h (kilometres per hour) [approximately 126.1 mph (miles per hour), 109.6 kn (knots), or 56.4 m/s (metres per second)][2] held by Richard Jenkins (UK) in *Greenbird* and set at Ivanpah Dry Lake, Prim, Nevada, USA on 26-Mar-2009. *Greenbird* was made entirely of carbon composite materials with a rigid wing sail and the only metal parts are the bearings for the sail and the wheels and the record was achieved in wind speeds of 48-65 km/h with a peak wind gust of 75 km/h (Borroz 2009, Dill 2009).

Team New Zealand, the *America's Cup* champions are set to challenge this record in mid-2022 with a carbon composite wing sail craft 14 m long, 7 m wide and 10 m high weighing 2.5 tonnes. The pilot will be the noted Olympic and America's Cup sailor Glenn Ashby and possible sites for the attempt are two dry salt lakes in Australia, Lake Gairdner in South Australia or Lake Lefroy in Western Australia (Johnstone 2022). Figure 1 shows a computer model of Team NZ's challenger

Figure 1

GPS Post Processed Kinematic (PPK) Data

Team NZ are proposing to use two survey-grade Leica Viva GS10 GPS receivers – one at a base station and the other onboard the land yacht – operating in differential kinematic mode recording carrier-phase measurements at 0.1 second intervals (10 Hz).

[1] The Global Positioning System (GPS), originally Navstar GPS, is a satellite-based radionavigation system owned by the United States government and operated by the United States Space Force. It is one of the global navigation satellite systems (GNSS) that provides geolocation and time information to a GPS receiver anywhere on or near the Earth where there is an unobstructed line of sight to four or more GPS satellites. (Wikipedia)

[2] Conversion factors are: 1 kn = 1.852 km/h (exactly), 1 mph = 1.609344 km/h (exactly), 3.6 km/h = 1 m/s.

Post-processing kinematic (PPK) data using Leica software yields 'baselines' that are straight lines in space whose terminal points are the receiver at the base station (stationary) and the receiver onboard the land yacht (moving). These baselines are defined by $\left(X,Y,Z\right)$ coordinate triplets related to the World Geodetic System 1984 (WGS84)[3] that is the reference system of GPS and the length of a baseline is $L=\sqrt{\left(\Delta X\right)^2+\left(\Delta Y\right)^2+\left(\Delta Z\right)^2}$ where $\Delta X = X_{Yacht} - X_{Base}$ and similarly for $\Delta Y, \Delta Z$. The PPK processing software also allows the baselines to be defined in a local horizon system $\left(E,N,U\right)$ or (east, north, up) where the U-axis is in the direction of the normal to the reference ellipsoid passing through the base station and the N-U plane is the meridian plane through the base station and the E-N plane is a local horizon plane at the elevation of the base station receiver. The 'geocentric' XYZ system and the 'local' ENU system are connected by a sequence of rotations followed by translations and the PPK processing software is capable of producing baselines defined by $\left(E,N,U\right)$ triplets and $L=\sqrt{\left(\Delta E\right)^2+\left(\Delta N\right)^2+\left(\Delta U\right)^2}$ where $\Delta E = E_{Yacht} - E_{Base}$ and similarly for $\Delta N, \Delta U$, and it useful for our purposes to note that on a dry salt lake, the base station receiver and the moving receiver will be close to the same elevation and baselines will have a very small ΔU component. It is also common for PPK processing software to convert $\left(X,Y,Z\right)$ coordinate triplets to $\left(\phi,\lambda,h\right)$ geographical triplets where ϕ is latitude, λ is longitude and h is ellipsoidal height and then transform the $\left(\phi,\lambda\right)$ geographical coordinates to $\left(E,N\right)$ coordinates on a Universal Transverse Mercator (UTM) projection and transform the ellipsoidal height into a height related to mean sea level.

If the receiver at the base station is considered as fixed, the PPK processing software can produce E,N,U coordinates of the moving receiver at fixed intervals of time where the time interval Δt is the data recording rate of 10 Hz (0.1 sec). As an example, Table 1 shows a portion of 28200 observations of a kinematic GPS survey where the roving receiver was located in the bow of a rowing boat (coxless 4) on Lake Burley Griffin

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Rowing Test Data
Coords from post-processed kinematic GPS survey
Data at 0.1 second epochs
Epoch 1 is Zone Time (11h E. of UT) 7h 35m 22.1s (27322.1s)
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Epoch	East (E)	North (N)	Elevation (U)
	m	m	m
1	691629.515	6092064.916	554.805
2	691629.502	6092064.931	554.819
3	691629.493	6092064.944	554.800
:	:	:	:
:	:	:	:
:	:	:	:
28198	690737.213	6091549.876	558.711
28199	690737.195	6091549.792	558.660
28200	690737.198	6091549.696	558.704

Table 1. Extract from PPK GPS data set

[3] The origin of WGS84 is located at the Earth's centre of mass and the Z-axis is the rotational axis of the Earth passing through the origin and the X-Y plane is the equatorial plane perpendicular to the Z-axis and containing the origin. Four parameters define a model Earth centered at the origin that is both a reference ellipsoid for position and also a level surface of a reference gravity field. The reference ellipsoid (an ellipse rotated about its minor axis) has an equatorial radius $a = 6378137$ m and flattening $f = 1/298.257223563$. The X-Z plane is the meridian of zero longitude (Greenwich meridian) and the positive X-axis passes through the intersection of meridian of zero longitude and the equator and the Y-axis is advanced 90 degrees east around the equator. Latitudes are measured from 0° to ±90° (positive north and negative south) and longitudes are measured from 0° to ±180° (positive east and negative west) of the zero meridian.

In this paper, we will discuss and test three methods of determining velocity of the land yacht from E_k, N_k coordinates derived from PPK GPS data at instants of time t_k for $k = 1,2,3,\ldots$ They are; (i) from time and coordinate differences obtained from successive PPK measurements and using the simple relationship that velocity is distance divided by time; (ii) from the land yacht's cumulative distance travelled s_k derived from E_k, N_k at times t_k and a first-order central difference approximation of velocity; and (iii) from a Kalman Filter using E_k, N_k at times t_k as the input measurement data.

To test the formula and procedures we will develop a data set derived from a velocity curve that is a function of time that we denote $v(t)$ and that this curve has the general shape of a *Logistic curve,* an s-shaped-curve that is asymptotic to two lines, one of which is a line denoting a maximum velocity and the other is the t-axis where velocity is zero. Once having determined a suitable function for $v(t)$ we can integrate with respect to time t and determine the function $s(t)$ where s is a distance along the velocity curve and then differentiate $v(t)$ with respect to t and determine the function $a(t)$ where a is the acceleration. We also differentiate acceleration to obtain a function $j(t)$ that is known as *jerk* (the rate of change of acceleration). This is a useful quantity in dynamic studies where force equals mass by acceleration $(F = ma)$ and if force is a function of time, say $F(t)$, and mass m is constant then a small change in force, perhaps modelled by the small increment δF , then by the *Total Increment Theorem* $\delta F = \frac{\partial F}{\partial t}\delta t = m\frac{da}{dt}\delta t$ where $\frac{da}{dt}$ is jerk.

Once having obtained a function $s(t)$ it is a simple operation to obtain distances along a horizontal line having a desired bearing (a clockwise angle from grid north) from an origin (say the stationary land yacht) and convert distances and bearing to coordinates E_k, N_k at times t_k. To simulate an actual run along a course on a flat salt lake, the computed coordinates will be disturbed by the addition of small random quantities drawn from a *Normal* distribution having a mean $\mu = 0$, variance $\sigma^2 = 0.0001\ \text{m}^2$ and standard deviation $\sigma = +\sqrt{\sigma^2} = 0.010\ \text{m}$ (Deakin 2005) where the notation $\sigma = +\sqrt{\sigma^2}$ indicates that σ is the positive square root of σ^2. We think this value of $\sigma = 0.010$ is reasonable – and it implies that distances of 10 km computed from these disturbed coordinates will have standard deviations of approximately 0.014 m. – if our test data is modelling a land yacht on a salt flat moving at approximately 200 kph (55 m/s). [The calculation of the value of 0.014 m is an exercise in Propagation of Variances (Deakin 2005) that will be dealt with in a following section]

The test data set simulating E_k, N_k coordinates at times t_k derived from PPK data will then be used to determine velocity of the land yacht in three ways: (i) from time and coordinate differences obtained from successive PPK measurements and using the simple relationship that velocity is distance divided by time; (ii) from the land yacht's cumulative distance travelled s_k derived from PPK measurements at times t_k for $k = 1,2,3,\ldots$ and a first-order central difference approximation of velocity; and (iii) from a Kalman Filter using local plane coordinates E_k, N_k derived from PPK measurements at times t_k.

The three methods are discussed in following sections.

In addition to a study of the methods of calculating velocity from PPK data we also assess meaningful ways of expressing a representative velocity over a fixed time interval. For example, the North American Land Sailing Association (NALSA) in their regulations for speed record attempts state "The top speed will be the average over three consecutive seconds." and "The primary method must have an accuracy of plus or minus 0.5 mph (0.8 km/hr) or less. Accuracy is defined as twice the combined measurement uncertainty of the measurement system (ie at 95% confidence)." (NALSA 2009)

[In this second passage ‘primary method’ means speed measurements taken in a scientifically valid method approved by the NALSA board.] We will show how averages (means) and standard deviations can be calculated from samples of derived velocity data and confidence intervals calculated for stated averages.

Generation of Data for Testing

To see how velocity can be calculated from PPK GPS data we can generate data of a fictitious land yacht accelerating from rest and reaching a steady velocity. Suppose the velocity/time curve of this land yacht has a shape akin to the curve of a *Logistic function* (Deakin 2018) whose general form can be given as

$$y = \frac{A_1 - A_2}{1 + e^{r(x - x_0)}} + A_2 \tag{1}$$

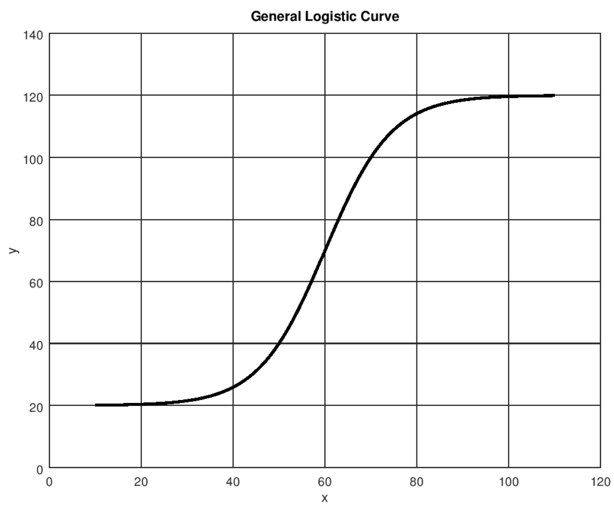

Figure 2. Logistic curve: $y = \dfrac{A_1 - A_2}{1 + e^{r(x - x_0)}} + A_2$, $A_1 = 20$, $A_2 = 120$, $r = 0.138629436$, $x_0 = 60$

$y = A_1$ is the lower asymptote of the curve given by (1) when $x \to -\infty$ and $y = A_2$ is the upper asymptote of the curve when $x \to +\infty$. The midpoint of the curve is $\left[x_0, \frac{1}{2}\left(A_1 + A_2\right)\right]$ when $x = x_0$ then $e^{r(x - x_0)} = e^0 = 1$ and $y = \frac{1}{2}\left(A_1 + A_2\right)$

For our purposes we would like $A_1 = 0$, $A_2 = A$ in (1) and replace x with t (time) and y with v (velocity) giving the Logistic function as $v = \dfrac{-A}{1 + e^{r(t - t_0)}} + A$ or $v = A\left(1 - \dfrac{1}{1 + e^{r(t - t_0)}}\right) = A\left(\dfrac{1 - \left(1 + e^{r(t - t_0)}\right)}{1 + e^{r(t - t_0)}}\right)$ and $v = \dfrac{A e^{r(t - t_0)}}{1 + e^{r(t - t_0)}}$. Multiplying numerator and denominator by $e^{-r(t - t_0)}$ and using the rule $e^a e^{-a} = e^{(a - a)} = e^0 = 1$ gives velocity as a function of time that is useful for our purposes

$$v(t) = \frac{A}{1 + e^{-r(t - t_0)}} \tag{2}$$

This s-shaped curve has $v = 0$ as the lower asymptote when $t \to -\infty$ and $v = A$ as the upper asymptote of the curve when $t \to +\infty$. The midpoint of the curve is $\left[t_0, \frac{1}{2}A\right]$ when $t = t_0$ then $e^{-r(t-t_0)} = e^0 = 1$ and $v = \frac{1}{2}A$.

To determine the constant r we may arbitrarily fix a value $v_0 = v(0)$ that is the velocity at $t = 0$ and (2) can be re-arranged as $e^{r t_0} = \left(A/v_0\right) - 1$ and taking logarithms of both sides gives

$$r = \frac{1}{t_0}\ln\left(\frac{A}{v_0} - 1\right) \tag{3}$$

The current speed record for a land yacht is 202.9 km/h or approximately 56.4 m/s (since 3.6 km/h = 1 m/s), so for our purposes we could set $A = 60$ m/s, $v_0 = 0.01$ m/s and $t_0 = 15$ sec.

Substituting these values into (3) gives $r = \dfrac{1}{15}\ln(5999) \approx 0.58$

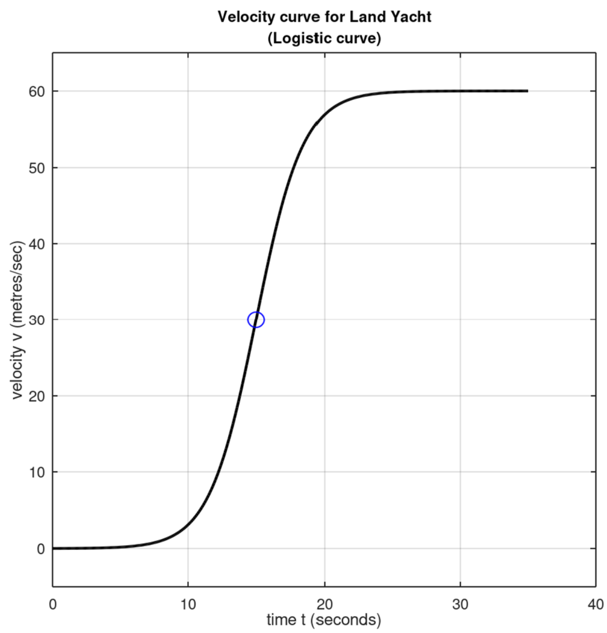

Figure 3. Velocity curve: $v(t) = \dfrac{A}{1 + e^{-r(t-t_0)}}$, $A = 60$, $t_0 = 15$, $v_0 = 0.01$, $r = \dfrac{1}{t_0}\ln\left(\dfrac{A}{v_0} - 1\right) \approx 0.58$

We have defined velocity $v(t) = \dot{s}(t) = \dfrac{ds}{dt}$ and acceleration $a(t) = \ddot{s}(t) = \dfrac{d^2s}{dt^2}$ where s is a distance along the velocity curve. Acceleration can also be expressed as $a(t) = \dfrac{dv}{dt}$ and if we differentiate (2) we obtain

$$a(t) = \frac{dv}{dt} = \frac{\left(1 + e^{-r(t-t_0)}\right)\dfrac{dA}{dt} - A\dfrac{d}{dt}\left(1 + e^{-r(t-t_0)}\right)}{\left(1 + e^{-r(t-t_0)}\right)^2} \qquad \text{quotient rule for derivatives}$$

$$= \frac{-A\dfrac{d}{du}\left(1 + e^{u}\right)\dfrac{du}{dt}}{1 + 2e^{-r(t-t_0)} + \left(e^{-r(t-t_0)}\right)^2} \qquad \text{with the substitution } u = -r(t - t_0)$$

and

$$a(t) = \frac{Ar\,e^{-r(t-t_0)}}{e^{-r(t-t_0)}\left(e^{r(t-t_0)} + 2 + e^{-r(t-t_0)}\right)} = \frac{Ar}{2 + e^{r(t-t_0)} + e^{-r(t-t_0)}}$$

and since $2\cosh x = e^{x} + e^{-x}$

$$a(t) = \frac{Ar}{2\left(1 + \cosh\left(r\left(t - t_0\right)\right)\right)} \tag{4}$$

And differentiating acceleration with respect to time gives jerk j and

$$j(t) = -\frac{Ar^2}{2}\left(\frac{\sinh\left(r\left(t - t_0\right)\right)}{\left(1 + \cosh\left(r\left(t - t_0\right)\right)\right)^2}\right) \tag{5}$$

If velocity $v(t) = \dot{s}(t) = \dfrac{ds}{dt}$ then an equation for a distance s along the velocity curve can be obtained by integration, i.e. $s(t) = \int v(t)\,dt$ and from (2) we have

$$s(t) = \int \frac{A\,dt}{1 + e^{-r(t-t_0)}} = A\int \frac{dt}{1 + e^{-rt}\,e^{rt_0}} = A\int \frac{dt}{1 + ce^{-rt}}$$

where $c = e^{rt_0}$ is a constant. Using the standard integral result $\int \dfrac{dx}{p + qe^{ax}} = \dfrac{x}{p} - \dfrac{1}{ap}\ln\left(p + qe^{ax}\right)$ we have $s(t) = A\left(t + \dfrac{1}{r}\ln\left(1 + ce^{-rt}\right) + C\right)$ where C is a constant of integration that can be evaluated by defining $s(t) = 0$ when $t = 0$ yielding $C = -\dfrac{1}{r}\ln\left(1 + c\right)$ and $s(t) = A\left(t + \dfrac{1}{r}\left(\ln\left(1 + ce^{-rt}\right) - \ln\left(1 + c\right)\right)\right)$.

Finally, using the rule for logarithms: $\ln A - \ln B = \ln\left(\dfrac{A}{B}\right)$ gives

$$s(t) = A\left(t + \frac{1}{r}\ln\left(\frac{1 + ce^{-rt}}{1 + c}\right)\right) \quad \text{for } t > 0 \tag{6}$$

where the constant $c = e^{rt_0}$

We now have the necessary equations to generate our data set.

First, we have generated Land Yacht Reference Data at 0.1 second intervals from $t = 0$ to 45 sec that is:

- (i) cumulative distance $s(t)$ using (6),
- (ii) velocity $v(t)$ using (2),
- (iii) acceleration $a(t)$ using (4),
- (iv) jerk using (5) and
- (v) East and North coordinates at distances s along a straight line bearing $\phi = 45^\circ$ where $\begin{Bmatrix} E \\ N \end{Bmatrix} = \begin{Bmatrix} E_0 + s\sin\phi \\ N_0 + s\cos\phi \end{Bmatrix}$ and E_0, N_0 are coordinates at $t = 0$

The constants for this reference data are: $A = 60$ m/s, $v_0 = 0.01$ m/s, $t_0 = 15$ sec.

Portions of the Land Yacht Reference Data are shown in Table 2.

Second, we have generated Simulated PPK Data (Epoch, East, North) by adding small random values from a *Normal* distribution with mean $\mu = 0$, variance $\sigma^2 = 0.0001\ \text{m}^2$ and standard deviation $\sigma = +\sqrt{\sigma^2} = 0.010\ \text{m}$ to the East and North coordinates of the Land Yacht Reference Data set and rounded the values to the nearest 0.001 m.

This simulated PPK data is shown in Appendix A and portions are shown in Table 3.

LAND YACHT REFERENCE DATA

Epoch	time	distance	velocity	accel'n	jerk	East	North
1	0.0	0.000000	0.010000	0.005799	0.003362	1000.000000	1000.000000
2	0.1	0.001030	0.010597	0.006145	0.003562	1000.000728	1000.000728
3	0.2	0.002121	0.011230	0.006511	0.003775	1000.001499	1000.001499
4	0.3	0.003277	0.011900	0.006900	0.004000	1000.002317	1000.002317
5	0.4	0.004502	0.012610	0.007312	0.004239	1000.003183	1000.003183
:	:	:	:	:	:	:	:
50	4.9	0.278033	0.171004	0.098892	0.057026	1000.196599	1000.196599
51	5.0	0.295637	0.181183	0.104761	0.060390	1000.209047	1000.209047
52	5.1	0.314290	0.191967	0.110977	0.063950	1000.222236	1000.222236
:	:	:	:	:	:	:	:
150	14.9	68.736501	29.130309	8.692037	0.146137	1048.604046	1048.604046
151	15.0	71.693010	30.000000	8.699348	-0.000000	1050.694614	1050.694614
152	15.1	74.736501	30.869691	8.692037	-0.146137	1052.846687	1052.846687
:	:	:	:	:	:	:	:
300	29.9	894.001030	59.989403	0.006145	-0.003562	1632.154190	1632.154190
301	30.0	900.000000	59.990000	0.005799	-0.003362	1636.396103	1636.396103
302	30.1	905.999028	59.990563	0.005472	-0.003173	1640.638057	1640.638057
:	:	:	:	:	:	:	:
449	44.8	1787.982759	59.999998	0.000001	-0.000001	2264.294734	2264.294734
450	44.9	1793.982759	59.999998	0.000001	-0.000001	2268.537374	2268.537374
451	45.0	1799.982759	59.999998	0.000001	-0.000001	2272.780015	2272.780015

Table 2. Land Yacht Reference Data at 0.1 second intervals.
(East, North coordinates are points on a line bearing 45°)

LAND YACHT SIMULATED PPK DATA

Epoch	East	North
1	999.979	999.992
2	1000.009	999.999
3	1000.010	999.988
4	1000.001	1000.000
5	1000.012	1000.004
:	:	:
50	1000.196	1000.217
51	1000.203	1000.200
52	1000.209	1000.223
:	:	:
150	1048.590	1048.604
151	1050.703	1050.683
152	1052.833	1052.835
:	:	:
300	1632.157	1632.158
301	1636.414	1636.394
302	1640.639	1640.626
:	:	:
449	2264.287	2264.292
450	2268.531	2268.539
451	2272.759	2272.772

Table 3. Land Yacht Simulated PPK Data at 0.1 second intervals.

A Simple Approximation of Velocity from GPS Positions

Suppose that the land yacht is stationary for a period of time prior to a speed run, then the PPK data (coordinates) will be relatively similar and as the land yacht's sail is adjusted to harness the wind energy it will accelerate away from the start until its velocity reaches a maximum for the given conditions. This acceleration will be reflected in increasing differences between coordinates at the regular intervals $\Delta t = 0.1$ sec that will gradually oscillate around certain values that reflect maximum velocity.

Of course, since velocity is distance divided by time, we could define coordinate differences $\Delta E_k = E_{k+1} - E_{k-1}$ and $\Delta N_k = N_{k+1} - N_{k-1}$ of observations at times t_{k-1}, t_{k+1} where $k = 1, 2, 3, \ldots$ and assume $\Delta U \approx 0$ because of the flat terrain; and approximations of the velocities in the east and north directions are

$$v_{E_k} = \frac{E_{k+1} - E_{k-1}}{t_{k+1} - t_{k-1}} = \frac{\Delta E_k}{2\Delta t}, \quad v_{N_k} = \frac{N_{k+1} - N_{k-1}}{t_{k+1} - t_{k-1}} = \frac{\Delta N_k}{2\Delta t} \quad \text{at time } t_k \tag{7}$$

and the velocity of the land yacht in the direction of travel is

$$v_k = \sqrt{\left(v_{E_k}\right)^2 + \left(v_{N_k}\right)^2} \quad \text{at time } t_k \tag{8}$$

Equations (7) are known as 1$^{\text{st}}$-order central difference approximations, where the term '1$^{\text{st}}$-order' relates to the number of data points either side of the central point of interest; in this case 1 point either side of the central point at time t_k. The following section has a more detailed treatment of central difference approximations.

We have used equations (7) and (8) to calculate approximations of velocity at regular intervals $\Delta t = 0.1$ sec using the Land Yacht Simulated PPK Data (see Appendix A) and compared these values with the reference velocity in the Land Yacht Reference Data. A plot of the results from $t = 24$ to 32 sec is shown below in Figure 4 and calculated values for $t = 26$ to 30 sec are shown in Table 4.

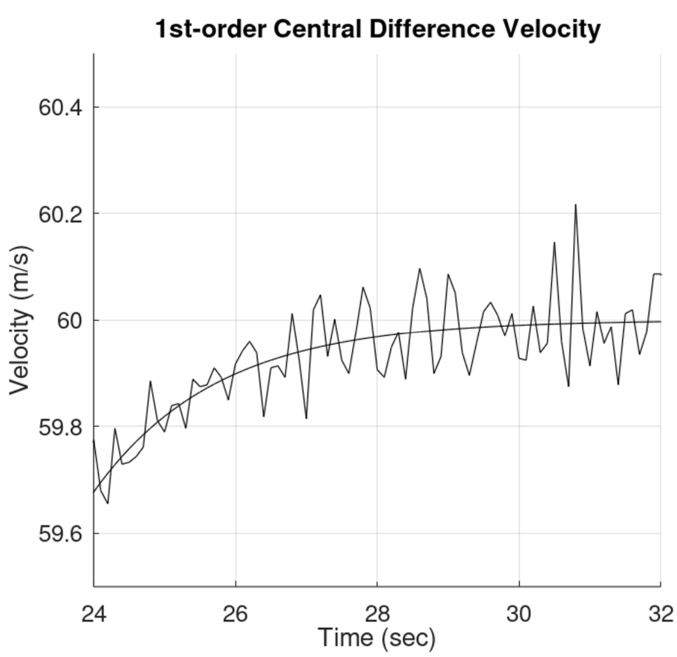

Figure 4. The irregular black line is velocity calculated using equations (7) and (8) with the Land Yacht Simulated PPK Data in Appendix A. The smooth curved line is the reference velocity.

Epoch	time	v1	vref	diff (v1-vref)	Epoch	time	v1	vref	diff (v1-vref)
261	26.0	59.917	59.898416	0.018584	282	28.1	59.892	59.969911	-0.077911
262	26.1	59.941	59.904131	0.036869	283	28.2	59.949	59.971605	-0.022605
263	26.2	59.959	59.909525	0.049475	284	28.3	59.977	59.973205	0.003795
264	26.3	59.938	59.914615	0.023385	285	28.4	59.888	59.974714	-0.086714
265	26.4	59.818	59.919420	-0.101420	286	28.5	60.023	59.976138	0.046862
266	26.5	59.910	59.923955	-0.013955	287	28.6	60.097	59.977482	0.119518
267	26.6	59.913	59.928234	-0.015234	288	28.7	60.040	59.978750	0.061250
268	26.7	59.892	59.932273	-0.040273	289	28.8	59.899	59.979947	-0.080947
269	26.8	60.012	59.936085	0.075915	290	28.9	59.931	59.981077	-0.050077
270	26.9	59.924	59.939683	-0.015683	291	29.0	60.086	59.982143	0.103857
271	27.0	59.814	59.943079	-0.129079	292	29.1	60.051	59.983149	0.067851
272	27.1	60.019	59.946283	0.072717	293	29.2	59.938	59.984098	-0.046098
273	27.2	60.048	59.949307	0.098693	294	29.3	59.895	59.984994	-0.089994
274	27.3	59.931	59.952161	-0.021161	295	29.4	59.956	59.985839	-0.029839
275	27.4	60.002	59.954855	0.047145	296	29.5	60.016	59.986637	0.029363
276	27.5	59.924	59.957397	-0.033397	297	29.6	60.033	59.987390	0.045610
277	27.6	59.899	59.959796	-0.060796	298	29.7	60.009	59.988100	0.020900
278	27.7	59.977	59.962059	0.014941	299	29.8	59.970	59.988770	-0.018770
279	27.8	60.062	59.964196	0.097804	300	29.9	60.012	59.989403	0.022597
280	27.9	60.023	59.966212	0.056788	301	30.0	59.927	59.990000	-0.063000
281	28.0	59.906	59.968115	-0.062115					

Table 4. Column 'v1' is the velocity computed using equations (7) and (8) with the Land Yacht Simulated PPK Data. Column 'vref' is the reference velocity from Land Yacht Reference Data and column 'diff' is the computed velocity – reference velocity.

Some Statistics from a Sample of the Computed Velocities

Velocities computed from the Land Yacht Simulated Data using the 1st-order central difference approximation (7) and (8) have a variability that is connected with the precision of the *E,N* coordinates and this variability can be seen in the irregular black line in Figure 4 as compared with the smooth line of reference velocities. We may assess the 'quality' of our computed velocity by using statistical measures of *variance, standard deviation* (the positive square root of variance) and *Root Mean Square* (RMS) and some description of these terms may be useful.

Mean, Variance, Standard Deviation, Root Mean Square

Variance σ_x^2 is a measure of dispersion of a population of *N* quantities $x_1, x_2, x_3, \ldots x_N$ about its mean value μ_x and

$$\mu_x = \frac{1}{N}\sum_{k=1}^{N} x_k \quad \text{and} \quad \sigma_x^2 = \frac{1}{N}\sum_{k=1}^{N}\left(x_k - \mu_x\right)^2 \quad \text{with } \sigma_x = +\sqrt{\sigma_x^2} \tag{9}$$

where the notation $q = +\sqrt{q^2}$ indicates that q is the positive square root of q^2.

Often, the total population of quantities is unknown (hence μ_x, σ_x^2 are unknown) and we only have samples of size *n* to estimate the population mean and variance, and these estimates are denoted $\overline{x}$ the sample mean, and s_x^2 the sample variance and

$$\overline{x} = \frac{1}{n}\sum_{k=1}^{n} x_k \quad \text{and} \quad s_x^2 = \frac{1}{n-1}\sum_{k=1}^{n}\left(x_k - \overline{x}\right)^2 \quad \text{with } s_x = +\sqrt{s_x^2} \tag{10}$$

μ_x, σ_x^2 (population of size *N*) and s_x, s_x^2 (sample of size *n*) are well known statistics of data $\left\{x_1, x_2, \ldots, x_N\right\}$ or $\left\{x_1, x_2, \ldots, x_n\right\}$.

Another statistical measure of a data set $\{x_1, x_2, \ldots, x_n\}$ is the Root Mean Square (RMS) and is defined as (Deakin & Kildea 1999)

$$\text{RMS} = +\sqrt{\frac{1}{n}\sum_{k=1}^{n}\left(x_k - a_k\right)^2} \tag{11}$$

where a_k refers to an accepted value for x_k. RMS is also known as the *quadratic mean* of a set of n numbers and when the accepted value in any sample is a (a constant) and the mean of the sample is $\bar{x}$ then (11) becomes

$$\left(\text{RMS}\right)^2 = \left(\frac{1}{n}\sum_{k=1}^{n}\left(x_k - \bar{x}\right)^2\right) + \left(\bar{x} - a\right)^2 \tag{12}$$

or in words

$$\left(\text{RMS}\right)^2 = \text{estimate of variance} + \left(\text{estimate of bias}\right)^2 \tag{13}$$

and *bias* in statistics is just another name for systematic error.

There are many instances (in the literature and in practice) where RMS is confused with standard deviation σ and they are only equivalent measures when the entire population is known (i.e., $n = N$) and the accepted value a_k is a constant value equal to the (population) mean μ.

It is interesting to note that Gauss (1821-28, p. 11) gave an equation for m^2 where he called m the *mean error* or *mean error to be feared* that matches (13)

In our sample of size $n = 41$ (see Table 4 above) the data we will deal with are the differences between the computed velocity and the reference velocity in the column head 'diff' and we denote these values as the set $\{x_1, x_2, x_3, \ldots, x_{41}\}$. Using equations (10) the following statistics are:

sample mean $\bar{x} = 0.001338$,
sample variance $s_x^2 = 0.003928$ and,
sample standard deviation $s_x = 0.062677$.

And the Root Mean Square (RMS) with the accepted value $a_k = 0$ as a constant in (11) is

$\text{RMS} = 0.061922$

If we use (12) and (13) an estimate of bias is given by: $\text{bias} = \bar{x} - a = \sqrt{\left(\text{RMS}\right)^2 - \left(\frac{n-1}{n}\right)s_x^2} = 0.001463$

and it would appear from the very small value of 0.001463 that there is no systematic error affecting our calculation method.

Confidence Intervals

We can use the mean and standard deviation to determine a *confidence interval* for describing a representative value of the data on the assumption that the data is random (i.e., does not contain systematic errors) and has the characteristics of a *probability density function* (pdf). It is common to assume *residuals* (small unknown random corrections to observed values) are normally distributed with mean μ, variance σ^2 and pdf $f\left(x\right) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ that is the familiar bell-shaped or Gaussian curve that is symmetric about $x = \mu$, asymptotic to the x-axis as $x \to \pm\infty$ and encloses an area of unity (see Figure 5).

The areas under the curve between the lines $x = \mu \pm \sigma$, $x = \mu \pm 2\sigma$ and $x = \mu \pm 3\sigma$ are 0.6827, 0.9545 and 0.9973 respectively. Alternatively, we may say that 68.27% of normally distributed random variables lie in the range $\mu \pm \sigma$, 95.45% in the range $\mu \pm 2\sigma$ and 99.73% in the range $\mu \pm 3\sigma$. And 95% of random variables lie in the range $\mu \pm 1.96\sigma$ which gives rise to the expression: the 95% Confidence Interval (CI) is $\pm 1.96\sigma$

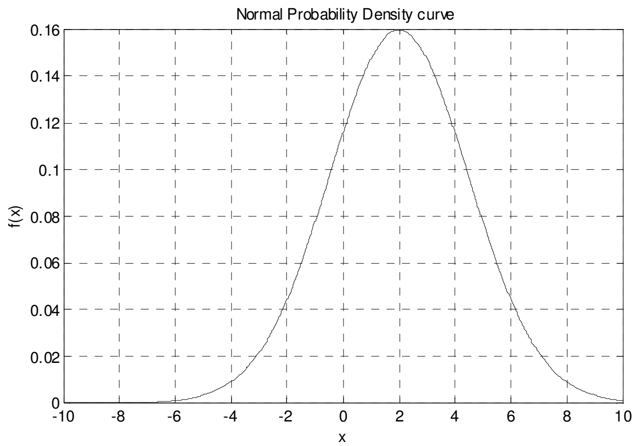

Figure 5. Curve of the Normal probability density function for $\mu = 2$ and $\sigma = 2.5$

In our sample of size $n = 41$ (see Table 4 above) the data we will deal with are the differences between the computed velocity and the reference velocity in the column head 'diff' and we denote these values as the set $\{x_1, x_2, x_3, \ldots, x_{41}\}$. Using the results above the sample statistics are:

sample mean $\bar{x} = 0.001338$,

sample variance $s_x^2 = 0.003928$ and,

sample standard deviation $s_x = 0.062677$.

and we may state the 95% Confidence Interval of the sample mean as $\pm 1.96 s_x = \pm(1.96 \times 0.062677)$ which is approximately ± 0.1228

Central Difference Approximations of Velocity from GPS Positions

In this section, we outline the method of deriving central difference approximation formula for computing *velocity* (rate of change of distance), *acceleration* (rate of change of velocity) and *jerk* (rate of change of acceleration) of a moving object whose instantaneous position is known at fixed and regular time intervals along its path of travel. These central difference approximations are in fact numerical differentiations of discrete functions of time and in the GPS literature, e.g. Bruton *et al.* (1999), a common approach is to differentiate position to give velocity, then differentiate velocity to give acceleration and differentiate acceleration to give jerk. In this section, we give central difference approximations for computing acceleration directly from position without the intermediate step of first computing velocity.

Kinematic GPS positions (E_k, N_k at instants of time t_k) can be converted to distances s_k measured along the path of the receiver from the start of the survey where $t_{START} = 0.0$ seconds and $s_{START} = 0.000$ metres. The distances s_k can be considered as discrete measurements of a continuous function of *t*, written as $s(t)$ and expanded into a series using Taylor's theorem

$$s(t) = s(t_k) + (t - t_k)\dot{s}(t_k) + \frac{(t - t_k)^2}{2!}\ddot{s}(t_k) + \frac{(t - t_k)^3}{3!}\dddot{s}(t_k) + \frac{(t - t_k)^4}{4!}\ddddot{s}(t_k) + \cdots + R_n \tag{14}$$

$s(t_k)$ is the function evaluated at time t_k, $\dot{s}(t_k)$ is the first derivative $\frac{ds}{dt}$ evaluated at t_k with higher order derivatives written as $\ddot{s}(t_k)$, $\dddot{s}(t_k)$, $\cdots$, etc and R_n is a remainder. The first derivative $\dot{s}(t_k)$ is the velocity $v(t_k)$ and the second derivative $\ddot{s}(t_k)$ is the acceleration $a(t_k)$ and the third derivative $\dddot{s}(t_k)$ is jerk.

Central difference approximations to derivatives are derived using an alternative form of the Taylor series obtained by letting $t = t_k + n\Delta t$ in equation (14) giving

$$s(t_k + n\Delta t) = s(t_k) + n\Delta t\,\dot{s}(t_k) + \frac{(n\Delta t)^2}{2!}\ddot{s}(t_k) + \frac{(n\Delta t)^3}{3!}\dddot{s}(t_k) + \frac{(n\Delta t)^4}{4!}\ddddot{s}(t_k) + \cdots \tag{15}$$

Letting n take pairs of values, say $(1,-1)$, $(2,-2)$, $(3,-3)$, etc in equation (15) and multiplying each resulting pair of equations by pairs of different coefficients, say $(h_1,-h_1)$, $(h_2,-h_2)$, $(h_3,-h_3)$, etc gives rise to sets of equations that when added together, with suitable values of the coefficients *h,* eliminate certain derivatives in the sum leaving higher order derivatives with increasingly smaller coefficients if Δt is small. Re-arranging the summation and ignoring the higher order terms give approximations to required derivatives. The term central difference arises from the fact that terms in the equations use observed values of the function $s(t)$ at times $\Delta t,\ 2\Delta t,\ 3\Delta t,$ etc either side of a 'central' time t_k.

For example, let $n = 1$ and then $n = -1$ in (15) giving two equations

$$\begin{aligned} s(t_k + \Delta t) &= s(t_k) + \Delta t\,\dot{s}(t_k) + \frac{(\Delta t)^2}{2!}\ddot{s}(t_k) + \frac{(\Delta t)^3}{3!}\dddot{s}(t_k) + \frac{(\Delta t)^4}{4!}\ddddot{s}(t_k) + \cdots \\ s(t_k - \Delta t) &= s(t_k) - \Delta t\,\dot{s}(t_k) + \frac{(\Delta t)^2}{2!}\ddot{s}(t_k) - \frac{(\Delta t)^3}{3!}\dddot{s}(t_k) + \frac{(\Delta t)^4}{4!}\ddddot{s}(t_k) - \cdots \end{aligned}$$

Now multiply the first equation by $h_1 = 1$ and the second equation by $-h_1 = -1$ giving

$$s(t_k + \Delta t) = s(t_k) + \Delta t\,\dot{s}(t_k) + \frac{(\Delta t)^2}{2!}\ddot{s}(t_k) + \frac{(\Delta t)^3}{3!}\dddot{s}(t_k) + \frac{(\Delta t)^4}{4!}\ddddot{s}(t_k) + \cdots \tag{i}$$

and $$-s(t_k - \Delta t) = -s(t_k) + \Delta t\,\dot{s}(t_k) - \frac{(\Delta t)^2}{2!}\ddot{s}(t_k) + \frac{(\Delta t)^3}{3!}\dddot{s}(t_k) - \frac{(\Delta t)^4}{4!}\ddddot{s}(t_k) + \cdots \tag{ii}$$

Adding (i) and (ii) gives

$$s(t_k + \Delta t) - s(t_k - \Delta t) = 2\Delta t\,\dot{s}(t_k) + \frac{(\Delta t)^3}{3}\dddot{s}(t_k) + \cdots \tag{iii}$$

Subtracting (ii) from (i) gives

$$s(t_k + \Delta t) + s(t_k - \Delta t) = 2s(t_k) + (\Delta t)^2\,\ddot{s}(t_k) + \frac{(\Delta t)^4}{12}\ddddot{s}(t_k) + \cdots \tag{iv}$$

and if Δt is small then $(\Delta t)^3$ in (iii) and $(\Delta t)^4$ in (iv) will be exceedingly small and may be ignored (along with higher-order terms) and rearrangements of (iii) and (iv) will give the 1st-order central difference approximations of velocity and acceleration as

$$\dot{s}(t_k) = \frac{s(t_k + \Delta t) - s(t_k - \Delta t)}{2\Delta t} \tag{16}$$

$$\ddot{s}\left(t_k\right) = \frac{s(t_k + \Delta t) + s(t_k - \Delta t) - 2s\left(t_k\right)}{\left(\Delta t\right)^2} \tag{17}$$

Central difference approximations to derivatives are known in the GPS literature as 1st , 2nd , 3rd -order etc. This nomenclature simply denotes the number of intervals about the central time and does not indicate the order of magnitude of neglected terms; which is the usual mathematical usage of the term order.

The usual method of deriving acceleration from GPS positions is to first obtain velocities $v_k = \dot{s}\left(t_k\right)$ from a central difference approximation and then treat these as discrete values of the continuous function of time $v\left(t\right)$. A second application of a central difference approximation (replacing s with v in relevant formula) yields the accelerations $a_k = \dot{v}\left(t_k\right)$. An alternative is to use approximations of the second derivative $a_k = \ddot{s}\left(t_k\right)$ as shown above.

Writing for velocity $v_k = \dot{s}\left(t_k\right)$ and for acceleration $a_k = \ddot{s}\left(t_k\right)$ at times t_k given distances $s_k = s\left(t_k\right), s_{k+1} = s\left(t_k + \Delta t\right), s_{k-1} = s\left(t_k - \Delta t\right)$, equations (16) and (17) can be written as

$$v_k = \frac{1}{2\Delta t}\left(s_{k+1} - s_{k-1}\right) \tag{18}$$

$$a_k = \frac{1}{\left(\Delta t\right)^2}\left(s_{k+1} + s_{k-1} - 2s_k\right) \tag{19}$$

Precision of velocity derived from 1st-order central differences

In this study we are assuming that the standard deviations of PPK derived coordinates E_k, N_k are 0.010 m and we denote these quantities as s_E, s_N. These are estimates of the (unknown) population standard deviations σ_E, σ_N and we note that standard deviations are positive square roots of variances s_E^2, s_N^2 (estimates) and σ_E^2, σ_N^2 (population). Furthermore, we will assume that E_k, N_k are independent random variables and their covariance $\sigma_{EN} = s_{EN} = 0$ where s_{EN} is an estimate of the covariance σ_{EN}.

To assist in the determination of precisions of derived (or computed) quantities we shall use the *Law of Propagation of Variances* that can be expressed in the following way (Deakin 2005)

> If $\mathbf{y} = f\left(\mathbf{x}\right)$ and $\mathbf{y}$ is an $\left(m,1\right)$ vector of quantities and $\mathbf{x}$ is an $\left(n,1\right)$ vector of non-linear functions of random variables then

$$\mathbf{\Sigma}_{yy} = \mathbf{J}_{yx}\mathbf{\Sigma}_{xx}\mathbf{J}_{yx}^T \tag{20}$$

where $\mathbf{\Sigma}_{yy}, \mathbf{\Sigma}_{xx}$ are square symmetric matrices containing variances on the leading diagonal and covariances on the off diagonal elements $\mathbf{\Sigma}_{yy} = \begin{bmatrix} \sigma_{y_1}^2 & \sigma_{y_1y_2} & \cdots & \sigma_{y_1y_m} \\ \sigma_{y_2y_2} & \sigma_{y_2}^2 & \cdots & \sigma_{y_2y_m} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{y_my_1} & \sigma_{y_my_2} & \cdots & \sigma_{y_m}^2 \end{bmatrix}, \mathbf{\Sigma}_{xx} = \begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1x_2} & \cdots & \sigma_{x_1x_n} \\ \sigma_{x_2x_2} & \sigma_{x_2}^2 & \cdots & \sigma_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{x_nx_1} & \sigma_{x_nx_2} & \cdots & \sigma_{x_n}^2 \end{bmatrix}$ and $\mathbf{J}_{yx}$ is an $\left(m,n\right)$ Jacobian matrix of partial derivatives and $\mathbf{J}_{yx} = \begin{bmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 & \cdots & \partial y_1/\partial x_n \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 & \cdots & \partial y_2/\partial x_n \\ \vdots & \vdots & & \vdots \\ \partial y_m/\partial x_1 & \partial y_m/\partial x_2 & \cdots & \partial y_m/\partial x_n \end{bmatrix}$

This rule also applies to cofactor matrices

If $\mathbf{y} = f(\mathbf{x})$ and $\mathbf{y}$ is an $(m,1)$ vector of quantities and $\mathbf{x}$ is an $(n,1)$ vector of non-linear functions of random variables then

$$\mathbf{Q}_{yy} = \mathbf{J}_{yx}\mathbf{Q}_{xx}\mathbf{J}_{yx}^T \tag{21}$$

where $\mathbf{Q}_{yy}, \mathbf{Q}_{xx}$ are square symmetric matrices containing estimates of variances and covariances and

$$\mathbf{Q}_{yy} = \begin{bmatrix} s_{y_1}^2 & s_{y_1y_2} & \cdots & s_{y_1y_m} \\ s_{y_2y_2} & s_{y_2}^2 & \cdots & s_{y_2y_m} \\ \vdots & \vdots & \ddots & \vdots \\ s_{y_my_1} & s_{y_my_2} & \cdots & s_{y_m}^2 \end{bmatrix}, \mathbf{Q}_{xx} = \begin{bmatrix} s_{x_1}^2 & s_{x_1x_2} & \cdots & s_{x_1x_n} \\ s_{x_2x_2} & s_{x_2}^2 & \cdots & s_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{x_nx_1} & s_{x_nx_2} & \cdots & s_{x_n}^2 \end{bmatrix}$$

The Law of Propagation of Variances is often expressed as an algebraic equation. For example, if z is a function of two random variables x and y, i.e., $z = f(x,y)$ then the variance of z is

$$\sigma_z^2 = \left(\frac{\partial z}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial z}{\partial y}\right)^2 \sigma_y^2 + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sigma_{xy} \tag{22}$$

Equation (22) can be derived from (20) in the following manner. Let $z = f(x,y)$ be written as $\mathbf{y} = f(\mathbf{x})$ where $\mathbf{y} = [z]$ is a $(1,1)$ vector and $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ is a $(2,1)$ vector. The variance matrix of the vector $\mathbf{x}$ is $\mathbf{\Sigma}_{xx} = \begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1x_2} \\ \sigma_{x_2x_2} & \sigma_{x_2}^2 \end{bmatrix}$, the Jacobian matrix $\mathbf{J}_{yx} = \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix}$ and the variance matrix $\mathbf{\Sigma}_{yy}$ contains the single element σ_z^2 given by

$$\mathbf{\Sigma}_{yy} = \left[\sigma_z^2\right] = \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix} \begin{bmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{bmatrix}$$

Expanding this equation gives (22)

In the case where the random variables in $\mathbf{x}$ are independent, i.e., their covariances are zero; we have the Special Law of Propagation of Variances. For the case of $z = f(x,y)$ where the random variables x and y are independent, the *Special Law of Propagation of Variances* is written as

$$\sigma_z^2 = \left(\frac{\partial z}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial z}{\partial y}\right)^2 \sigma_y^2 \tag{23}$$

Now, consider a distance s_{ij} between two points P_i, P_j whose coordinates are E_i, N_i, E_j, N_j and

$$s_{ij} = \sqrt{\left(E_j - E_i\right)^2 + \left(N_j - N_i\right)^2}$$

The distance s_{ij} is a function of the four random variables E_i, N_i, E_j, N_j and if we consider that the variables are independent and their covariances are zero then the Special Law of Propagation of Variances is

$$\sigma_{s_{ij}}^2 = \left(\frac{\partial s_{ij}}{\partial E_j}\right)^2 \sigma_{E_j}^2 + \left(\frac{\partial s_{ij}}{\partial N_j}\right)^2 \sigma_{N_j}^2 + \left(\frac{\partial s_{ij}}{\partial E_i}\right)^2 \sigma_{E_i}^2 + \left(\frac{\partial s_{ij}}{\partial N_i}\right)^2 \sigma_{N_i}^2$$

and the partial derivatives are (Deakin 2005)

$$\frac{\partial s_{ij}}{\partial E_j} = \frac{E_j - E_i}{s_{ij}} = \sin\phi_{ij}, \quad \frac{\partial s_{ij}}{\partial N_j} = \frac{N_j - N_i}{s_{ij}} = \cos\phi_{ij},$$
$$\frac{\partial s_{ij}}{\partial E_i} = \frac{-\left(E_j - E_i\right)}{s_{ij}} = -\sin\phi_{ij}, \quad \frac{\partial s_{ij}}{\partial N_i} = \frac{-\left(N_j - N_i\right)}{s_{ij}} = -\cos\phi_{ij}$$

where ϕ is an angle measured clockwise from the north axis. Now, if the variances are considered to be equal, i.e., $\sigma^2_{E_i} = \sigma^2_{E_j} = \sigma^2_{N_i} = \sigma^2_{N_j} = \sigma^2_C$ then the variance of the distance s_{ij} is

$$\sigma^2_{s_{ij}} = 2\left(\sin^2\phi + \cos^2\phi\right)\sigma^2_C = 2\,\sigma^2_C$$

where σ^2_C is the variance of PPK derived coordinates.

We will now use this result in the 1st-order central difference formula (18) and write

$$v_k = \frac{1}{2\Delta t}\left(s_{k+1} - s_{k-1}\right) = \frac{s_{ij}}{2\Delta t}$$

where $s_{ij} = s_{k+1} - s_{k-1}$. If the time interval Δt is considered as a constant, then the velocity is a function of the distance s_{ij} and the variance of v at time t_k is given by

$$\sigma^2_v = \left(\frac{\partial v}{\partial s_{ij}}\right)^2 \sigma^2_{s_{ij}} = \left(\frac{1}{2\left(\Delta t\right)}\right)^2 2\,\sigma^2_C = \frac{\sigma^2_C}{2\left(\Delta t\right)^2}$$

and

$$\sigma_v = \frac{\sigma_C}{\sqrt{2}\left(\Delta t\right)} \tag{24}$$

So, if $\Delta t = 0.1$ sec and we assume $\sigma_C = 0.010$ m then the standard deviation of the computed velocity is $\sigma_v = 0.0707$ m/s $= 0.2546$ km/h since 3.6 km/h = 1 m/s.

Suppose that we wish that the standard deviation of v be 0.1 km/h = 0.0278 m/s then rearranging (24) gives

$$\Delta t = \frac{\sigma_C}{\sigma_v\sqrt{2}} = \frac{0.010}{0.0278\sqrt{2}} = 0.2546 \approx 0.2 \text{ sec}$$

Similarly to before we have used equation (18) with $\Delta t = 0.2$ sec to calculate approximations of velocity at regular intervals using the Land Yacht Simulated PPK Data (Appendix A) and compared these values with the reference velocity in the Land Yacht Reference Data. A plot of the results from $t = 24$ to 32 sec is shown below in Figure 6

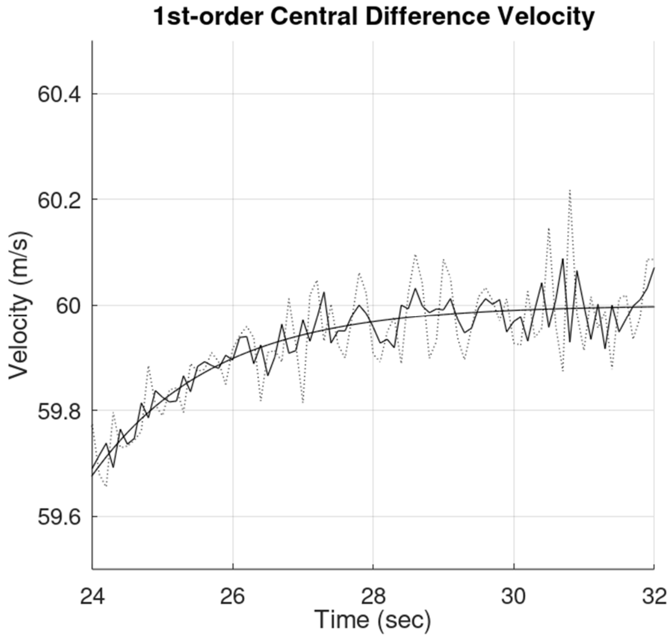

Figure 6. The irregular solid line is velocity calculated using equation (18) with $\Delta t = 0.2$ sec . The smooth curve is the reference velocity and the irregular dotted line is velocity calculated using equations (7) and (8) (see Figure 4)

Comparing the two computed velocities shown in Figure 6 it would appear that the variation from the reference velocity is much less for the solid line than for the dotted line which would indicate a more precise determination of the velocity using the methods of this section.

Epoch	time	v2	vref	diff (v2-vref)
261	26.0	59.895	59.898416	-0.003416
262	26.1	59.938	59.904131	0.033869
263	26.2	59.940	59.909525	0.030475
264	26.3	59.889	59.914615	-0.025615
265	26.4	59.924	59.919420	0.004580
266	26.5	59.866	59.923955	-0.057955
267	26.6	59.901	59.928234	-0.027234
268	26.7	59.963	59.932273	0.030727
269	26.8	59.908	59.936085	-0.028085
270	26.9	59.913	59.939683	-0.026683
271	27.0	59.972	59.943079	0.028921
272	27.1	59.931	59.946283	-0.015283
273	27.2	59.975	59.949307	0.025693
274	27.3	60.025	59.952161	0.072839
275	27.4	59.928	59.954855	-0.026855
276	27.5	59.951	59.957397	-0.006397
277	27.6	59.951	59.959796	-0.008796
278	27.7	59.981	59.962059	0.018941
279	27.8	60.000	59.964196	0.035804
280	27.9	59.984	59.966212	0.017788
281	28.0	59.958	59.968115	-0.010115

Epoch	time	v2	vref	diff (v2-vref)
282	28.1	59.927	59.969911	-0.042911
283	28.2	59.935	59.971605	-0.036605
284	28.3	59.919	59.973205	-0.054205
285	28.4	60.000	59.974714	0.025286
286	28.5	59.993	59.976138	0.016862
287	28.6	60.032	59.977482	0.054518
288	28.7	59.998	59.978750	0.019250
289	28.8	59.986	59.979947	0.006053
290	28.9	59.993	59.981077	0.011923
291	29.0	59.991	59.982143	0.008857
292	29.1	60.012	59.983149	0.028851
293	29.2	59.973	59.984098	-0.011098
294	29.3	59.947	59.984994	-0.037994
295	29.4	59.956	59.985839	-0.029839
296	29.5	59.995	59.986637	0.008363
297	29.6	60.012	59.987390	0.024610
298	29.7	60.002	59.988100	0.013900
299	29.8	60.010	59.988770	0.021230
300	29.9	59.949	59.989403	-0.040403
301	30.0	59.968	59.990000	-0.022000

Table 5. Column 'v2' is the velocity computed using equation (18) with $\Delta t = 0.2$ sec with the Land Yacht Simulated PPK Data. Column 'vref' is the reference velocity from Land Yacht Reference Data and column 'diff' is the computed velocity – reference velocity.

Mean, Variance, Standard Deviation of the sample from t = 26 to 30 sec.

In our sample of size $n = 41$ (see Table 5 above) the data we will deal with are the differences between the computed velocity and the reference velocity in the column head 'diff' and we denote these values as the set $\{x_1, x_2, x_3, \ldots, x_{41}\}$. Using equations (10) the following statistics are:

sample mean $\bar{x} = 0.000679$,

sample variance $s_x^2 = 0.000915$ and,

sample standard deviation $s_x = 0.030247$.

Comparing the sample standard deviation $s_x = 0.030247$ with the sample standard deviation from the previous method ($s_x = 0.062677$) indicates that this method is significantly more precise, as expected.

Velocity and acceleration from a Kalman Filter of GPS Positions

A Kalman filter is a set of mathematical equations written in matrix form that are applied recursively to estimate the *state* of a dynamic system. In our case, the dynamic system is the land yacht (with GPS receiver) moving across a dry salt lake. It receives position at time t_{k-1} that is East and North coordinates (E_{k-1}, N_{k-1}) from kinematic GPS measurements (the *primary* measurement model), and moves to position t_k according to a *dynamic* model, where it receives new position information. The state of the system at t_k is its position E_k, N_k and its velocity $\dot{E}_k, \dot{N}_k$. A Kalman filter takes into account the precisions of the measurements and the dynamic model and provides an efficient (recursive) computational solution to a least squares estimate of the state. That is, if the true values of the measurements are the observed values plus small unknown corrections (residuals) and the dynamic model has residuals accounting for the difference between theory and practice, then a least squares solution provides estimates that make the sum of the squared residuals (multiplied by weight coefficients) a minimum value. The weight of an observation is a measure of its precision.

The Kalman filter equations were published in 1960 by Dr. R.E. Kalman in his famous paper describing a new approach to the solution of linear filtering and prediction (Kalman 1960). Since that time, papers on the application of the technique have been filling numerous scientific journals and it is regarded as one of the most important algorithmic techniques ever devised. It has been used in applications ranging from navigating the Ranger and Apollo spacecraft in their lunar missions to predicting short-term fluctuations in the stock market. Sorenson (1970) shows Kalman's technique to be an extension of C.F. Gauss' original method of least squares developed in 1795 and provides an historical commentary on its practical solution of linear filtering problems studied by 20th century mathematicians.

The derivation of the Kalman filter equations can be found in many textbooks related to signal processing that is the usual domain of Electrical Engineers, e.g., Brown and Hwang (1992). These derivations often use terminology that is unfamiliar to surveyors, but two authors, Krakiwsky (1975) and Cross (1992) both with geodesy and surveying backgrounds, have derivations, explanations, terminology and examples that would be familiar to any surveyor. Deakin (2015) has a complete derivation of the Kalman filter equations with some worked examples and we used paragraphs from Deakin (2006, 2015) in the explanation above. Appendix B has a detailed explanation of the operation of the Kalman filter we are using for this project and we have written a program in GNU Octave[4] `Land_Yacht_Kalman` to process the data.

[4] GNU OCTAVE is free software featuring a high-level programming language, primarily intended for numerical computations that is mostly compatible with MATLAB. It is part of the GNU Project and is free software under the terms of the GNU General Public License.

The dynamic model we are using links the state (position and velocity) at times t_{k-1} and t_k according to the simple dynamic equations

$$\begin{aligned} E_k &= E_{k-1} + \dot{E}_{k-1}\Delta t + \tfrac{1}{2}\ddot{E}_{k-1}\left(\Delta t\right)^2 \\ N_k &= N_{k-1} + \dot{N}_{k-1}\Delta t + \tfrac{1}{2}\ddot{N}_{k-1}\left(\Delta t\right)^2 \end{aligned}$$

where time derivatives $\dot{E} = \dfrac{dE}{dt}$ and $\ddot{E} = \dfrac{d\dot{E}}{dt} = \dfrac{d^2E}{dt^2}$ are east velocity and acceleration respectively and similarly for $\dot{N}, \ddot{N}$ and $\Delta t = t_k - t_{k-1} = 0.1$ sec.

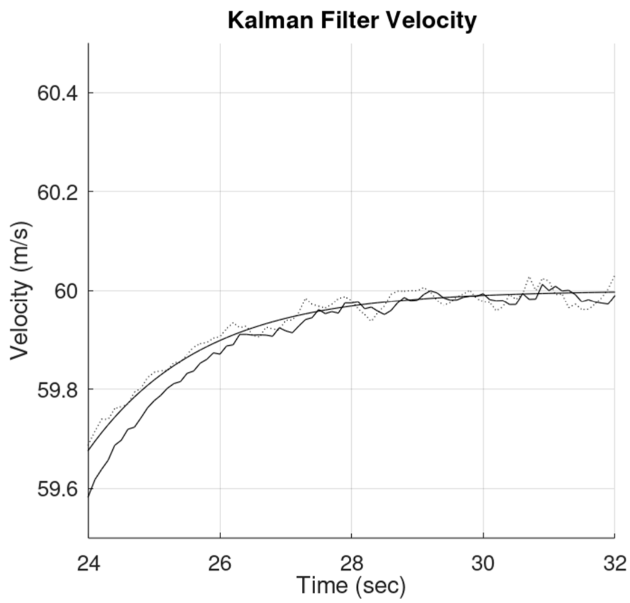

Figure 7. The irregular solid line is the Kalman filter velocity, the smooth curve is the reference velocity and the irregular dotted line is velocity calculated using equation (18) with $\Delta t = 0.2$ sec (see the solid line in Figure 6).

A portion of the output from Octave program `Land_Yacht_Kalman` is shown below for epochs 289 – 292 where $t_{289} = 28.8$ sec and $t_{292} = 29.1$ sec

```
Epoch = 289
Filtered State  Corrns       Filtered State cofactor matrix Qxx
   1585.498      0.001       0.000036  0.000000  0.000080  0.000000
   1585.498      0.006       0.000000  0.000036  0.000000  0.000080
     42.414      0.003       0.000080  0.000000  0.000400  0.000000
     42.419      0.013       0.000000  0.000080  0.000000  0.000400

Epoch = 290
Filtered State  Corrns       Filtered State cofactor matrix Qxx
   1589.738     -0.002       0.000036  0.000000  0.000080  0.000000
   1589.737     -0.002       0.000000  0.000036  0.000000  0.000080
     42.409     -0.005       0.000080  0.000000  0.000400  0.000000
     42.413     -0.005       0.000000  0.000080  0.000000  0.000400
```

```
Epoch = 291
Filtered State  Corrns       Filtered State cofactor matrix Qxx
   1593.982      0.003       0.000036  0.000000  0.000080  0.000000
   1593.976     -0.002       0.000000  0.000036  0.000000  0.000080
     42.417      0.007       0.000080  0.000000  0.000400  0.000000
     42.408     -0.005       0.000000  0.000080  0.000000  0.000400

Epoch = 292
Filtered State  Corrns       Filtered State cofactor matrix Qxx
   1598.229      0.005       0.000036  0.000000  0.000080  0.000000
   1598.220      0.003       0.000000  0.000036  0.000000  0.000080
     42.428      0.011       0.000080  0.000000  0.000400  0.000000
     42.414      0.006       0.000000  0.000080  0.000000  0.000400
```

At Epoch 289, the filtered state vector $\hat{\mathbf{x}}$ and filtered state cofactor matrix $\mathbf{Q}_{xx}$ are

$$\hat{\mathbf{x}}_{289} = \begin{bmatrix} E = 1585.498 \text{ m} \\ N = 1585.498 \text{ m} \\ \dot{E} = 42.414 \text{ m/s} \\ \dot{N} = 42.419 \text{ m/s} \end{bmatrix}_{289}, \quad \mathbf{Q}_{xx} = \begin{bmatrix} s_E^2 & s_{EN} & s_{E\dot{E}} & s_{E\dot{N}} \\ s_{NE} & s_N^2 & s_{N\dot{E}} & s_{N\dot{N}} \\ s_{\dot{E}E} & s_{\dot{E}N} & s_{\dot{E}}^2 & s_{\dot{E}\dot{N}} \\ s_{\dot{N}E} & s_{\dot{N}N} & s_{\dot{N}\dot{E}} & s_{\dot{N}}^2 \end{bmatrix}_{289} = \begin{bmatrix} 0.000036 & 0 & 0.000080 & 0 \\ 0 & 0.000036 & 0 & 0.000080 \\ 0.000080 & 0 & 0.000400 & 0 \\ 0 & 0.000080 & 0 & 0.000400 \end{bmatrix}_{289}$$

and the velocity $v_{289} = \sqrt{\left(\dot{E}_{289}\right)^2 + \left(\dot{N}_{289}\right)^2} = 59.986 \text{ m/s}$.

We may apply Propagation of Variances to the formula for velocity to obtain an estimate of the variance of the velocity in the following way. Since v is a function of velocities $\dot{E}, \dot{N}$ then

$s_v^2 = \left(\frac{\partial v}{\partial \dot{E}}\right)^2 s_{\dot{E}}^2 + \left(\frac{\partial v}{\partial \dot{N}}\right)^2 s_{\dot{N}}^2 + 2\left(\frac{\partial v}{\partial \dot{E}}\right)\left(\frac{\partial v}{\partial \dot{N}}\right) s_{\dot{E}\dot{N}}$ where the partial derivatives $\left(\frac{\partial v}{\partial \dot{E}}\right) = \frac{\dot{E}}{v}$ and $\left(\frac{\partial v}{\partial \dot{N}}\right) = \frac{\dot{N}}{v}$.

And since the covariance $s_{\dot{E}\dot{N}} = s_{\dot{N}\dot{E}} = 0$ then the estimate of the variance is

$$s_v^2 = \left(\frac{\dot{E}}{v}\right)^2 s_{\dot{E}}^2 + \left(\frac{\dot{N}}{v}\right)^2 s_{\dot{N}}^2 \tag{25}$$

Using the values for Epoch 289 gives $s_v^2 = 0.000400$ and $s_v = 0.020$ m/s

This demonstrates one of the advantages of using a Kalman filter to process the PPK GPS data: you get estimates of the precision of the elements of the state vector as a byproduct (the state cofactor matrix) of the filtering process.

The Average Velocity of 3 Consecutive Seconds of 10 Hz data

The proposed attempt by Emirates Team New Zealand to break the current speed record for a land yacht will use PPK GPS position data E_k, N_k at times t_k where $\Delta t = t_k - t_{k-1} = 0.1$ sec (10 Hz data rate) to compute velocity by the three methods outlined above; (i) from time and coordinate differences obtained from successive PPK measurements and using the simple relationship that velocity is distance divided by time using equations (7) and (8); (ii) from the land yacht's cumulative distance travelled s_k derived from E_k, N_k at times t_k and a first-order central difference approximation of velocity equation (18) with $\Delta t = 0.2$ sec; and (iii) from a Kalman Filter using E_k, N_k at times t_k as the input measurement data.

We have shown these three methods of computation using simulated PPK GPS data and from the accompanying diagrams it is clear that computed velocity has a variability due to random errors of measurement. In an actual speed run of a land yacht, it is likely that the computed velocity will have greater variability than our simulations and this would be due to random errors of measurement combined with errors induced by environmental factors (e.g., gusting winds, salt lake surface conditions, visibility, etc.) and mechanical/aerodynamic factors (e.g., wheel adhesion, steering, wing-sail orientation, etc.).

Any claim of a new speed record must be accompanied by 'evidence' of the land yacht exceeding the previous record speed and in previous record attempts this speed has been taken as the average speed over three consecutive seconds, noting that at the current record speed of 202.9 km/h a land yacht will cover approximately 170 m in 3 seconds (since 3.6 km/h = 1 m/s). And at a data rate of 10 Hz (PPK derived coordinates every 0.1 sec) the average speed will be computed from approximately 30 calculated velocities.

Tim Daddo Speed Test

To see how this will work in practice, one of the authors (Tim Daddo) conducted a speed test using a motor vehicle and two Leica Viva GS10 GPS receivers recording data at a rate of 10 Hz, one on the roof of the vehicle and the other at a base station. The PPK data at 698 epochs consisted of East and North coordinates, orthometric height, 2D CQ (coordinates) and 1D CQ (height) with Zone Time of epoch 1 = 19^h 51^m 04.3^s (Zone Time = UTC + 11^h). UTC is Coordinated Universal Time (formerly known as Greenwich Mean Time) and CQ is coordinate quality that is taken to mean the estimated standard deviation of the observations. A Kalman Filter (Octave program `Tim_Daddo_Kalman.m`) was used to process the data and a plot of velocity (speed) is shown in Figure 8.

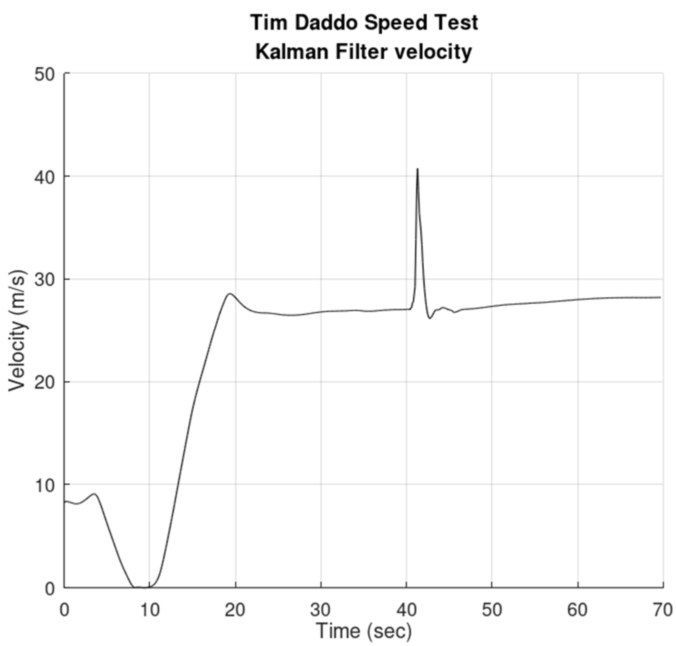

Figure 8. The solid line is the Kalman filter velocity (with data problem in the period between 40 and 42 seconds).

The spike in the velocity curve indicates a problem with the data in the period between 40 and 42 seconds. We are concerned with the 3 second period from 32 to 35 seconds and an extract of the output from Octave program `Tim_Daddo_Kalman.m` is shown below in Table 6 below.

For the 3 second period from 32 to 35 seconds the vehicle travelled approximately 81 m on fairly level terrain (orthometric height ranging between 75.807 m and 74.463 m) and the velocity was approximately 27 m/s = 97.2 km/h. During this period the 2D CQ ranged between 0.012 m and 0.007 m.

The Octave program computes estimates of the state of the system (the vehicle) at t_k that are its position E_k, N_k and its velocity $\dot{E}_k, \dot{N}_k$ in the coordinate directions, and from these the velocity v_k of the vehicle is calculated using $v_k = \sqrt{\left(\dot{E}_k\right)^2 + \left(\dot{N}_k\right)^2}$. These values are shown in Table 6 in the column headed 'v' and are called Kalman Filter velocities.

This file is: C:\Temp\Tim Daddo speed test 18Mar2022.out
Velocities from Kalman Filter (v) and
first-order central difference approximations (v1) and (v2)

Epoch	Time	East	North	s	v	v1	v2
1	0.0	321018.119	5754091.786	0.000	8.292		
2	0.1	321017.283	5754091.712	0.839	8.301	8.404	
3	0.2	321016.445	5754091.636	1.681	8.395	8.379	8.370
4	0.3	321015.614	5754091.564	2.515	8.375	8.337	8.324
5	0.4	321014.784	5754091.495	3.348	8.360	8.269	8.288
:	:	:	:	:	:	:	:
321	32.0	320524.202	5753888.600	536.254	26.901	26.906	26.901
322	32.1	320521.763	5753887.466	538.943	26.904	26.888	26.897
323	32.2	320519.320	5753886.344	541.632	26.904	26.888	26.917
324	32.3	320516.882	5753885.210	544.321	26.903	26.946	26.909
325	32.4	320514.431	5753884.077	547.021	26.910	26.929	26.928
326	32.5	320511.995	5753882.945	549.707	26.910	26.910	26.904
327	32.6	320509.552	5753881.805	552.403	26.915	26.878	26.911
328	32.7	320507.111	5753880.701	555.082	26.908	26.912	26.929
329	32.8	320504.666	5753879.548	557.785	26.913	26.980	26.929
330	32.9	320502.217	5753878.427	560.478	26.918	26.944	26.963
331	33.0	320499.770	5753877.296	563.174	26.924	26.946	26.944
332	33.1	320497.324	5753876.168	565.868	26.929	26.944	26.954
333	33.2	320494.877	5753875.038	568.563	26.934	26.962	26.958
334	33.3	320492.428	5753873.908	571.260	26.940	26.972	26.958
335	33.4	320489.980	5753872.776	573.957	26.947	26.954	26.961
336	33.5	320487.534	5753871.648	576.651	26.950	26.951	26.962
337	33.6	320485.086	5753870.517	579.348	26.954	26.970	26.955
338	33.7	320482.633	5753869.396	582.045	26.957	26.959	26.961
339	33.8	320480.186	5753868.268	584.739	26.959	26.952	26.969
340	33.9	320477.738	5753867.139	587.435	26.960	26.978	26.969
341	34.0	320475.285	5753866.009	590.135	26.963	26.985	26.955
342	34.1	320472.831	5753864.890	592.832	26.966	26.932	26.950
343	34.2	320470.391	5753863.761	595.522	26.962	26.914	26.940
344	34.3	320467.942	5753862.639	598.215	26.958	26.947	26.915
345	34.4	320465.497	5753861.502	600.911	26.955	26.916	26.923
346	34.5	320463.054	5753860.382	603.598	26.946	26.899	26.916
347	34.6	320460.609	5753859.257	606.291	26.939	26.915	26.902
348	34.7	320458.171	5753858.117	608.981	26.932	26.905	26.883
349	34.8	320455.735	5753856.977	611.672	26.925	26.852	26.855
350	34.9	320453.306	5753855.844	614.352	26.911	26.805	26.864
351	35.0	320450.874	5753854.715	617.033	26.893	26.877	26.872
:	:	:	:	:	:	:	:
405	40.4	320320.401	5753789.835	762.771	27.028	28.212	27.225
406	40.5	320317.765	5753788.461	765.743	27.218	27.709	28.869
407	40.6	320315.515	5753787.219	768.313	27.219	29.523	28.326
408	40.7	320312.554	5753785.685	771.648	27.709	28.939	30.100
409	40.8	320310.402	5753784.507	774.102	27.769	30.673	31.629
410	40.9	320307.133	5753782.814	777.783	28.531	34.280	46.462
411	41.0	320304.458	5753781.090	780.965	29.273	60.729	55.093
412	41.1	320300.102	5753772.910	790.233	33.847	75.237	52.803
413	41.2	320296.059	5753768.605	796.139	38.983	39.516	49.753
414	41.3	320293.307	5753768.874	798.904	40.767	21.748	33.988
415	41.4	320292.111	5753770.430	800.866	38.590	20.651	30.072
:	:	:	:	:	:	:	:
695	69.4	319665.259	5753315.935	1578.338	28.220	28.222	28.228
696	69.5	319663.118	5753314.097	1581.159	28.221	28.229	28.227
697	69.6	319660.972	5753312.262	1583.983	28.224	28.232	
698	69.7	319658.827	5753310.427	1586.806	28.225		

Table 6. Part of the output from Octave program `Tim_Daddo_Kalman.m`

The program computes two other velocities that are shown in the columns headed 'v1' and 'v2' and these are 1st-order central difference approximations v1 using equations (7) and (8), and v2 using equation (18) with cumulative distance s_k (see the column headed 's') and $\Delta t = 0.2$ sec.

Two plots of these velocities are shown in Figures 9 and 10. The first shows a plot of Kalman Filter velocity v and 1st-order central difference velocity v2 and the second shows a plot of v1 and v2

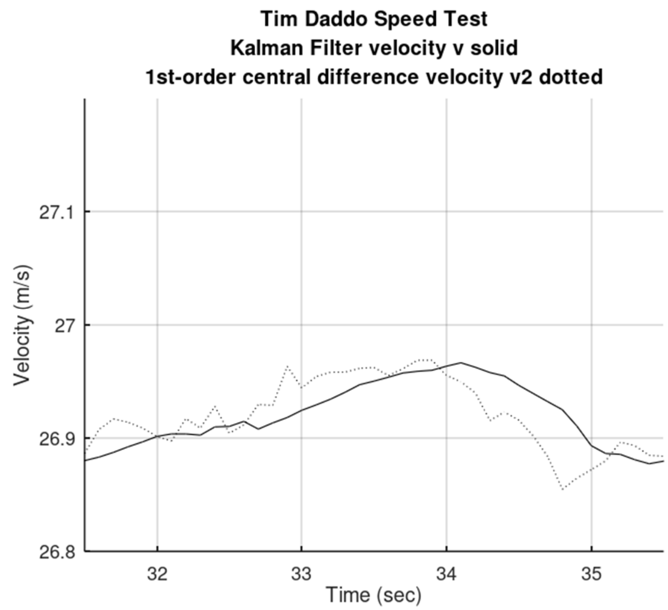

Figure 9. The solid line is the Kalman Filter velocity and the dotted line is the 1st-order approximation v2.

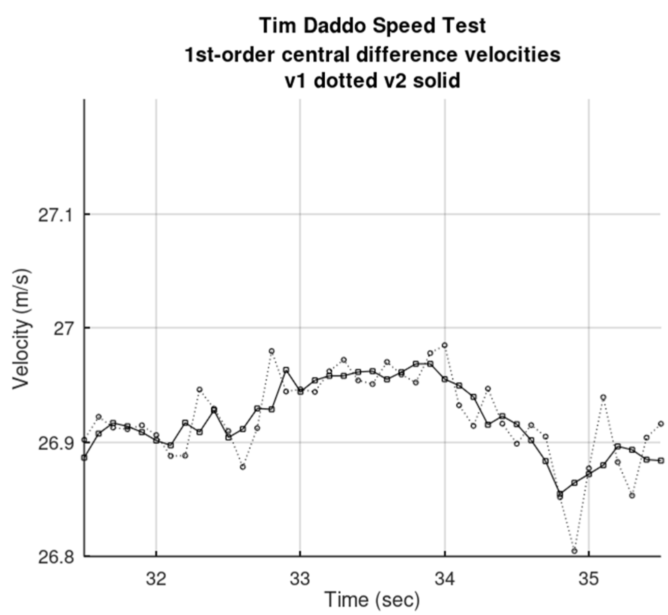

Figure 10. The solid line (and squares) is the 1st-order approximation v2 and the dotted line (with circles) is the 1st-order approximation v1.

The average velocities (with standard deviations and 95% Confidence Intervals) over the 3-second period from 32 to 35 seconds are

	velocities (m/s)		
	v	v1	v2
mean	26.932	26.926	26.928
standard deviation	0.023	0.041	0.032
95% CI	0.045	0.080	0.063

Note here that the 3-second period is from epoch 321 ($t = 32.0$ sec) to epoch 351 ($t = 35$ sec) and including the end-points there are 31 velocities.

A Moving Average Filter as an aid to velocity determination

To determine time periods to be used in calculating an average velocity over a defined interval, say 3 seconds as in the example above, it may be useful to employ a *Moving Average Filter*.

Suppose that our velocity data are the $n = 698$ Kalman Filter velocities from our vehicle trial, the Tim Daddo Speed Test (see the column headed 'v' in Table 6) and we denote these as the ordered set $\left\{ v_1 \quad v_2 \quad v_3 \quad \cdots \quad v_{697} \quad v_{698} \right\}$ where the subscripts are the epoch numbers E increasing from 1 to 698.

We choose a 'window' of period (or width) p that is superimposed over our ordered set and the average of the values in the window is calculated and denoted A_E. Then the window is moved one place to the right over the ordered set and a new average calculated and denoted A_{E+1} and this process repeated for averages A_{E+2}, A_{E+3}, etc. In our case we choose that the initial average is associated with the right-hand end of the window and our *right moving average* is given by

$$A_E = \frac{1}{p} \sum_{j=1-p}^{0} v_{E+j} \quad \text{for } E = p, p+1, p+2, \ldots, n \tag{26}$$

If we choose $p = 31$, since there are 31 velocities in a 3-second period (including the end-points) then the sequence of averages A_E given by (26) would be $\left\{ A_{31} \quad A_{32} \quad A_{33} \quad \cdots \quad A_{697} \quad A_{698} \right\}$ where $A_{31} = \frac{1}{31}\left(v_{31} + v_{30} + \cdots + v_1 \right)$, $A_{32} = \frac{1}{31}\left(v_{32} + v_{31} + \cdots + v_2 \right)$, etc. and we could say that the averages A_E are *filtered* values that may or may not be close to the actual velocity for that particular epoch E. By studying the sequence of calculations, we may write

$$A_E = A_{E-1} + \frac{1}{p}\left(v_E - v_{E-p} \right) \tag{27}$$

and only the initial average needs to be calculated.

Figure 11 shows the Kalman Filter velocities for the Tim Daddo Speed Test from $t = 25$ sec to $t = 40$ sec as a solid line and the Right Moving Average (3-second period) as a dotted line.

A Moving Average Filter could be a useful tool to determine which periods of a speed trial should be used to determine an appropriate average speed in a record attempt.

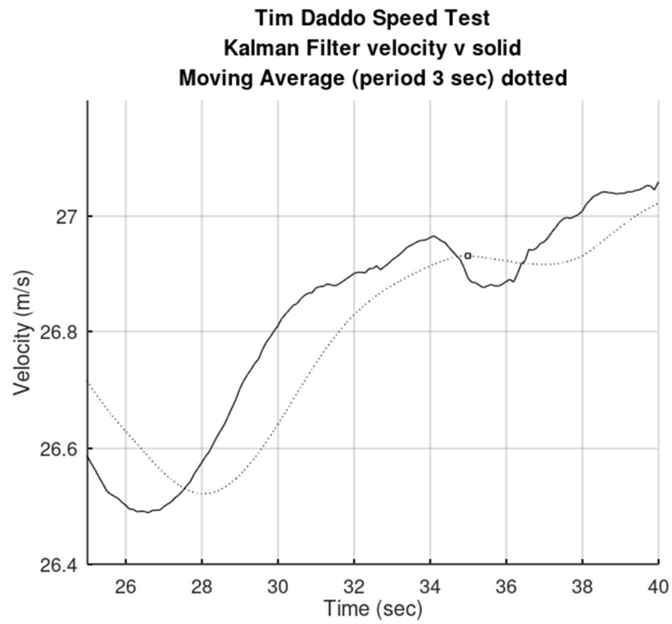

Figure 11. The solid line is the Kalman Filter velocity and the dotted line is the Right Moving Average (with period 3 sec). The small square is the moving average $A_{351} = 26.932$ m/s at $t = 35.0$ sec.

Conclusion

In this study we have discussed and derived formula for three methods of determining velocity of a land yacht from E_k, N_k coordinates derived from PPK GPS data at instants of time t_k for $k = 1,2,3,\ldots$ They are; (i) from time and coordinate differences obtained from successive PPK measurements and using the simple relationship that velocity is distance divided by time; (ii) from the land yacht's cumulative distance travelled s_k derived from E_k, N_k at times t_k and a first-order central difference approximation of velocity; and (iii) from a Kalman Filter using E_k, N_k at times t_k as the input measurement data. We have tested the formula using a simulated data set (by adding small random values from a normal probability distribution to reference data related to a logistic function) and we are confident our formula gives reasonable estimates of velocity. And we have provided formula (and examples) for the calculation of the statistics mean, variance, standard deviation and root-mean-square (RMS) of samples of calculated velocities, as well as some information and formula for determining 95% confidence intervals of calculated sample means.

In addition to the calculation of results and statistics we have shown how the Law of Propagation of Variances can be employed to determine an estimate of the precision of a computed velocity and this allows us to compare the expected precisions of calculated velocities from the three different methods.

In an actual speed trial with a land yacht the governing body FISLY[5] requires several continuous seconds of calculated velocities to demonstrate a land yacht has achieved a certain velocity. We have simulated a speed trial using a motor vehicle (the Tim Daddo Speed Test) and have shown how our three methods yield velocity data at 0.1 second intervals over a 3-second period. Our three methods give averages (mean results) that are very close (a range of 0.006 m/s) with standard deviations between 0.02 and 0.04 m/s and the 95% confidence intervals of the means are less than or equal to 0.08 m/s.

These are all acceptable results in our opinion and if there was one method to be favoured then it would be the method easiest to 'program'. And this would be our method of 1[st]-order central differences with

[5] Federation International de Sand at Land Yachting

$\Delta t = 0.2$ sec – shown as 'v2' in our examples above. This method is very easy to implement on Microsoft's Excel spreadsheet software.

As an aid to determining which time period of a speed trial would be worthy of inspection we have also provided some information (and an example) of a Moving Average Filter.

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APPENDIX A

Simulated PPK Data for Land Yacht

Epoch	East	North	Epoch	East	North	Epoch	East	North
1	999.979	999.992	59	1000.333	1000.335	117	1009.516	1009.524
2	1000.009	999.999	60	1000.379	1000.367	118	1010.054	1010.050
3	1000.010	999.988	61	1000.387	1000.392	119	1010.622	1010.619
4	1000.001	1000.000	62	1000.412	1000.402	120	1011.190	1011.219
5	1000.012	1000.004	63	1000.439	1000.419	121	1011.823	1011.811
6	1000.001	999.997	64	1000.456	1000.460	122	1012.472	1012.466
7	1000.025	1000.015	65	1000.496	1000.481	123	1013.147	1013.157
8	1000.005	999.990	66	1000.517	1000.536	124	1013.861	1013.874
9	1000.002	1000.009	67	1000.532	1000.554	125	1014.607	1014.634
10	1000.001	1000.008	68	1000.569	1000.585	126	1015.388	1015.409
11	1000.016	1000.012	69	1000.617	1000.613	127	1016.241	1016.216
12	999.996	1000.021	70	1000.648	1000.657	128	1017.092	1017.090
13	1000.007	1000.009	71	1000.689	1000.672	129	1018.000	1018.007
14	1000.023	1000.013	72	1000.737	1000.729	130	1018.941	1018.936
15	999.996	1000.011	73	1000.769	1000.770	131	1019.934	1019.945
16	1000.017	1000.002	74	1000.810	1000.830	132	1020.966	1020.975
17	1000.018	1000.010	75	1000.863	1000.863	133	1022.058	1022.054
18	1000.030	1000.007	76	1000.934	1000.920	134	1023.185	1023.180
19	1000.022	1000.008	77	1000.978	1001.008	135	1024.355	1024.363
20	1000.024	1000.023	78	1001.048	1001.038	136	1025.576	1025.585
21	1000.024	1000.031	79	1001.115	1001.112	137	1026.861	1026.861
22	1000.031	1000.041	80	1001.161	1001.162	138	1028.187	1028.195
23	1000.037	1000.034	81	1001.243	1001.255	139	1029.579	1029.575
24	1000.031	1000.030	82	1001.303	1001.311	140	1031.016	1031.018
25	1000.032	1000.003	83	1001.416	1001.399	141	1032.514	1032.508
26	1000.033	1000.049	84	1001.479	1001.484	142	1034.056	1034.069
27	1000.042	1000.027	85	1001.556	1001.564	143	1035.681	1035.686
28	1000.061	1000.038	86	1001.645	1001.662	144	1037.346	1037.332
29	1000.036	1000.028	87	1001.756	1001.760	145	1039.076	1039.058
30	1000.056	1000.050	88	1001.876	1001.861	146	1040.864	1040.856
31	1000.060	1000.056	89	1001.967	1001.988	147	1042.697	1042.699
32	1000.057	1000.064	90	1002.082	1002.099	148	1044.627	1044.605
33	1000.082	1000.059	91	1002.219	1002.202	149	1046.580	1046.598
34	1000.069	1000.049	92	1002.359	1002.339	150	1048.590	1048.604
35	1000.082	1000.073	93	1002.491	1002.487	151	1050.703	1050.683
36	1000.076	1000.081	94	1002.600	1002.625	152	1052.833	1052.835
37	1000.080	1000.084	95	1002.783	1002.767	153	1055.055	1055.061
38	1000.068	1000.092	96	1002.920	1002.932	154	1057.324	1057.331
39	1000.095	1000.102	97	1003.113	1003.123	155	1059.663	1059.661
40	1000.094	1000.117	98	1003.298	1003.289	156	1062.062	1062.071
41	1000.106	1000.113	99	1003.468	1003.476	157	1064.521	1064.516
42	1000.121	1000.129	100	1003.689	1003.667	158	1067.031	1067.027
43	1000.130	1000.138	101	1003.917	1003.904	159	1069.614	1069.641
44	1000.133	1000.119	102	1004.142	1004.125	160	1072.260	1072.259
45	1000.140	1000.162	103	1004.374	1004.377	161	1074.955	1074.932
46	1000.166	1000.152	104	1004.629	1004.627	162	1077.680	1077.680
47	1000.159	1000.155	105	1004.871	1004.906	163	1080.502	1080.515
48	1000.165	1000.180	106	1005.197	1005.168	164	1083.350	1083.334
49	1000.172	1000.181	107	1005.468	1005.480	165	1086.264	1086.264
50	1000.196	1000.217	108	1005.802	1005.791	166	1089.237	1089.225
51	1000.203	1000.200	109	1006.131	1006.121	167	1092.236	1092.236
52	1000.209	1000.223	110	1006.470	1006.470	168	1095.287	1095.303
53	1000.218	1000.223	111	1006.849	1006.839	169	1098.421	1098.429
54	1000.248	1000.260	112	1007.230	1007.240	170	1101.591	1101.578
55	1000.281	1000.288	113	1007.643	1007.632	171	1104.795	1104.790
56	1000.278	1000.273	114	1008.083	1008.076	172	1108.040	1108.057
57	1000.301	1000.295	115	1008.531	1008.552	173	1111.347	1111.335
58	1000.312	1000.315	116	1009.019	1009.010	174	1114.698	1114.662

Epoch	East	North	Epoch	East	North	Epoch	East	North
175	1118.057	1118.052	239	1373.769	1373.779	303	1644.865	1644.892
176	1121.465	1121.483	240	1377.991	1377.987	304	1649.126	1649.117
177	1124.939	1124.925	241	1382.216	1382.218	305	1653.365	1653.345
178	1128.424	1128.428	242	1386.442	1386.443	306	1657.595	1657.606
179	1131.959	1131.956	243	1390.656	1390.658	307	1661.861	1661.861
180	1135.505	1135.500	244	1394.874	1394.884	308	1666.077	1666.083
181	1139.088	1139.098	245	1399.116	1399.111	309	1670.325	1670.332
182	1142.718	1142.722	246	1403.328	1403.324	310	1674.595	1674.597
183	1146.404	1146.396	247	1407.569	1407.553	311	1678.812	1678.811
184	1150.071	1150.061	248	1411.777	1411.773	312	1683.071	1683.067
185	1153.784	1153.767	249	1416.000	1416.025	313	1687.298	1687.300
186	1157.496	1157.500	250	1420.252	1420.236	314	1691.543	1691.553
187	1161.271	1161.258	251	1424.470	1424.472	315	1695.786	1695.779
188	1165.056	1165.057	252	1428.688	1428.711	316	1700.017	1700.015
189	1168.861	1168.872	253	1432.943	1432.924	317	1704.272	1704.267
190	1172.701	1172.696	254	1437.169	1437.156	318	1708.505	1708.503
191	1176.548	1176.546	255	1441.391	1441.389	319	1712.740	1712.751
192	1180.410	1180.418	256	1445.634	1445.630	320	1716.977	1716.995
193	1184.311	1184.325	257	1449.844	1449.871	321	1721.229	1721.257
194	1188.220	1188.228	258	1454.098	1454.102	322	1725.481	1725.486
195	1192.152	1192.148	259	1458.321	1458.339	323	1729.737	1729.735
196	1196.111	1196.089	260	1462.563	1462.577	324	1733.980	1733.973
197	1200.075	1200.064	261	1466.787	1466.801	325	1738.212	1738.219
198	1204.048	1204.027	262	1471.042	1471.045	326	1742.439	1742.443
199	1208.038	1208.041	263	1475.267	1475.275	327	1746.703	1746.701
200	1212.019	1212.050	264	1479.530	1479.516	328	1750.950	1750.931
201	1216.037	1216.047	265	1483.747	1483.748	329	1755.197	1755.189
202	1220.059	1220.077	266	1487.974	1487.991	330	1759.436	1759.403
203	1224.103	1224.115	267	1492.210	1492.230	331	1763.646	1763.674
204	1228.176	1228.152	268	1496.469	1496.442	332	1767.928	1767.918
205	1232.210	1232.196	269	1500.687	1500.693	333	1772.164	1772.144
206	1236.276	1236.295	270	1504.946	1504.939	334	1776.409	1776.379
207	1240.349	1240.379	271	1509.166	1509.163	335	1780.626	1780.640
208	1244.465	1244.446	272	1513.403	1513.400	336	1784.874	1784.863
209	1248.533	1248.569	273	1517.653	1517.652	337	1789.101	1789.121
210	1252.635	1252.650	274	1521.892	1521.895	338	1793.354	1793.371
211	1256.775	1256.763	275	1526.138	1526.118	339	1797.595	1797.603
212	1260.881	1260.881	276	1530.368	1530.390	340	1801.854	1801.843
213	1265.005	1265.016	277	1534.597	1534.608	341	1806.092	1806.086
214	1269.144	1269.145	278	1538.850	1538.850	342	1810.337	1810.334
215	1273.287	1273.289	279	1543.077	1543.092	343	1814.573	1814.575
216	1277.435	1277.415	280	1547.355	1547.333	344	1818.820	1818.831
217	1281.583	1281.580	281	1551.583	1551.563	345	1823.042	1823.067
218	1285.728	1285.752	282	1555.810	1555.822	346	1827.304	1827.304
219	1289.900	1289.914	283	1560.044	1560.042	347	1831.536	1831.559
220	1294.050	1294.059	284	1564.300	1564.288	348	1835.783	1835.793
221	1298.205	1298.231	285	1568.520	1568.530	349	1840.031	1840.033
222	1302.392	1302.397	286	1572.772	1572.755	350	1844.271	1844.267
223	1306.576	1306.565	287	1577.006	1577.021	351	1848.525	1848.523
224	1310.768	1310.738	288	1581.270	1581.255	352	1852.765	1852.769
225	1314.931	1314.960	289	1585.501	1585.508	353	1857.002	1857.005
226	1319.135	1319.125	290	1589.734	1589.733	354	1861.241	1861.247
227	1323.317	1323.299	291	1593.988	1593.972	355	1865.484	1865.488
228	1327.521	1327.510	292	1598.238	1598.224	356	1869.733	1869.742
229	1331.695	1331.699	293	1602.471	1602.474	357	1873.968	1873.976
230	1335.897	1335.904	294	1606.694	1606.721	358	1878.228	1878.209
231	1340.089	1340.106	295	1610.935	1610.951	359	1882.460	1882.451
232	1344.310	1344.314	296	1615.188	1615.185	360	1886.696	1886.703
233	1348.499	1348.509	297	1619.433	1619.428	361	1890.926	1890.930
234	1352.720	1352.706	298	1623.684	1623.669	362	1895.173	1895.182
235	1356.928	1356.926	299	1627.927	1627.907	363	1899.422	1899.420
236	1361.134	1361.145	300	1632.157	1632.158	364	1903.663	1903.676
237	1365.359	1365.353	301	1636.414	1636.394	365	1907.930	1907.916
238	1369.567	1369.552	302	1640.639	1640.626	366	1912.153	1912.144

Epoch	East	North
367	1916.410	1916.397
368	1920.640	1920.645
369	1924.886	1924.868
370	1929.124	1929.118
371	1933.349	1933.372
372	1937.602	1937.607
373	1941.867	1941.846
374	1946.121	1946.075
375	1950.345	1950.346
376	1954.581	1954.559
377	1958.840	1958.821
378	1963.081	1963.071
379	1967.303	1967.300
380	1971.550	1971.514
381	1975.822	1975.809
382	1980.029	1980.034
383	1984.290	1984.278
384	1988.518	1988.522
385	1992.761	1992.767
386	1996.997	1997.006
387	2001.241	2001.261
388	2005.496	2005.492
389	2009.728	2009.730
390	2013.980	2013.987
391	2018.217	2018.227
392	2022.480	2022.478
393	2026.699	2026.694
394	2030.951	2030.953
395	2035.187	2035.201
396	2039.438	2039.430
397	2043.683	2043.688
398	2047.920	2047.912
399	2052.164	2052.180
400	2056.402	2056.385
401	2060.649	2060.650
402	2064.891	2064.897
403	2069.129	2069.131
404	2073.375	2073.384
405	2077.626	2077.615
406	2081.854	2081.852
407	2086.114	2086.116
408	2090.335	2090.330
409	2094.603	2094.575
410	2098.827	2098.850
411	2103.059	2103.073
412	2107.329	2107.305
413	2111.562	2111.541
414	2115.795	2115.807
415	2120.059	2120.044
416	2124.292	2124.279
417	2128.524	2128.516
418	2132.781	2132.767
419	2137.024	2137.022
420	2141.252	2141.251
421	2145.504	2145.497
422	2149.751	2149.739
423	2153.994	2153.994
424	2158.228	2158.229
425	2162.464	2162.464
426	2166.705	2166.710
427	2170.938	2170.968
428	2175.202	2175.198
429	2179.454	2179.425
430	2183.687	2183.689
431	2187.914	2187.929
432	2192.175	2192.173
433	2196.407	2196.419
434	2200.650	2200.657
435	2204.889	2204.911
436	2209.145	2209.142
437	2213.395	2213.365
438	2217.635	2217.614
439	2221.878	2221.865
440	2226.111	2226.114
441	2230.355	2230.345
442	2234.600	2234.597
443	2238.827	2238.833
444	2243.076	2243.077
445	2247.323	2247.322
446	2251.570	2251.553
447	2255.812	2255.813
448	2260.052	2260.067
449	2264.287	2264.292
450	2268.531	2268.539
451	2272.759	2272.772

APPENDIX B

The Kalman Filter

A Kalman filter is a set of mathematical equations that are applied recursively to estimate the *state* of a *dynamic system*. In our case, the dynamic system is the land yacht (with GPS receiver) moving over the dry salt lake and its state is its position, velocity and acceleration at instants of time t_k for $k = 1,2,3,\ldots$ To assist in the determination of the state, a *dynamic model* of the system is required and we propose a model that is extremely simple and often used in navigation problems and can be expressed as $y = y\left(t\right)$ that can be expanded about the point $t = t_k$ using Taylor's theorem

$$y\left(t\right) = y\left(t_k\right) + \left(t - t_k\right)\dot{y}\left(t_k\right) + \frac{\left(t - t_k\right)^2}{2!}\ddot{y}\left(t_k\right) + + \frac{\left(t - t_k\right)^3}{3!}\dddot{y}\left(t_k\right) + \cdots + R_n$$

where $\dot{y}\left(t_k\right), \ddot{y}\left(t_k\right), \dddot{y}\left(t_k\right),$ etc. are first, second, third, etc. derivatives with respect to t evaluated at t_k and R_n is the remainder after n terms. Letting $t = t_k + \Delta t$ and then $\Delta t = t - t_k$ we may write

$$y\left(t_k + \Delta t\right) = y\left(t_k\right) + \dot{y}\left(t_k\right)\Delta t + \frac{1}{2}\ddot{y}\left(t_k\right)\left(\Delta t\right)^2 + \frac{1}{6}\dddot{y}\left(t_k\right)\left(\Delta t\right)^3 + \cdots \tag{28}$$

We now have a power series expression for the continuous function $y\left(t\right)$ at the point $t = t_k + \Delta t$ involving the function y and its derivatives $\dot{y}, \ddot{y},$ etc. (all evaluated at t_k) and the time difference Δt.

In a similar manner, if we assume $\dot{y}\left(t_k\right), \ddot{y}\left(t_k\right),$ etc. to be continuous functions of *t,* then

$$\dot{y}\left(t_k + \Delta t\right) = \dot{y}\left(t_k\right) + \ddot{y}\left(t_k\right)\Delta t + \frac{1}{2}\dddot{y}\left(t_k\right)\left(\Delta t\right)^2 + \cdots$$
$$\ddot{y}\left(t_k + \Delta t\right) = \ddot{y}\left(t_k\right) + \dddot{y}\left(t_k\right)\Delta t + \cdots \tag{29}$$
etc.

Now considering two epochs of time t_k and t_{k-1} separated by a time interval Δt, we can combine equations (28) and (29) with a change of subscripts for t into dynamic models having the general matrix forms

$$\begin{gathered} \begin{bmatrix} y \\ \dot{y} \end{bmatrix}_k = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix}_{k-1} + \begin{bmatrix} \frac{1}{2}\left(\Delta t\right)^2 \\ 1 \end{bmatrix} \left[\ddot{y}\right]_{k-1} \\ \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix}_k = \begin{bmatrix} 1 & \Delta t & \frac{1}{2}\left(\Delta t\right)^2 \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix}_{k-1} + \begin{bmatrix} \frac{1}{6}\left(\Delta t\right)^3 \\ \frac{1}{2}\left(\Delta t\right)^2 \\ \Delta t \end{bmatrix} \left[\dddot{y}\right]_{k-1} \end{gathered} \tag{30}$$

If we have $y\left(t\right) = \left\{E, N\right\}\left(t\right)$ where E,N are east and north coordinates and $y\left(t\right) = \left\{\quad\right\}\left(t\right)$ means y is a function of time t where the function contains the variables within the braces $\left\{\quad\right\}$, then the derivatives are *velocity* $\dot{y}\left(t\right) = \left\{\dot{E}, \dot{N}\right\}\left(t\right)$, *acceleration* $\ddot{y}\left(t\right) = \left\{\ddot{E}, \ddot{N}\right\}\left(t\right)$ and *jerk* $\dddot{y}\left(t\right) = \left\{\dddot{E}, \dddot{N}\right\}\left(t\right)$ which is the rate of change of acceleration, and (30) can be written as the dynamic models

$$\begin{bmatrix} E \\ N \\ \dot{E} \\ \dot{N} \end{bmatrix}_k = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E \\ N \\ \dot{E} \\ \dot{N} \end{bmatrix}_{k-1} + \begin{bmatrix} \frac{1}{2}\left(\Delta t\right)^2 & 0 \\ 0 & \frac{1}{2}\left(\Delta t\right)^2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{E} \\ \ddot{N} \end{bmatrix}_{k-1} \tag{31}$$

and

$$\begin{bmatrix} E \\ N \\ \dot{E} \\ \dot{N} \\ \ddot{E} \\ \ddot{N} \end{bmatrix}_k = \begin{bmatrix} 1 & 0 & \Delta t & 0 & \frac{1}{2}\left(\Delta t\right)^2 & 0 \\ 0 & 1 & 0 & \Delta t & 0 & \frac{1}{2}\left(\Delta t\right)^2 \\ 0 & 0 & 1 & 0 & \Delta t & 0 \\ 0 & 0 & 0 & 1 & 0 & \Delta t \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E \\ N \\ \dot{E} \\ \dot{N} \\ \ddot{E} \\ \ddot{N} \end{bmatrix}_{k-1} + \begin{bmatrix} \frac{1}{6}\left(\Delta t\right)^3 & 0 \\ 0 & \frac{1}{6}\left(\Delta t\right)^3 \\ \frac{1}{2}\left(\Delta t\right)^2 & 0 \\ 0 & \frac{1}{2}\left(\Delta t\right)^2 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix} \begin{bmatrix} \dddot{E} \\ \dddot{N} \end{bmatrix}_{k-1} \tag{32}$$

These are the models that we will use in our case where the dynamic system is the land yacht (with GPS receiver) moving along a course on a dry salt lake, and it receives position at time t_{k-1}, that are East and North coordinates E_{k-1}, N_{k-1} from kinematic GPS measurements (the *primary* measurement model), and moves to position t_k, according to the *dynamic* models (31) or (32), where it receives new position information. We express these dynamic models in the following form

$$\hat{\mathbf{x}}_k = \mathbf{T}\mathbf{x}_{k-1} + \mathbf{v}_m \tag{33}$$

where

$$\mathbf{v}_m = \mathbf{H}\mathbf{w} \tag{34}$$

- $\hat{\mathbf{x}}_k$ and $\mathbf{x}_{k-1}$ are $\left(u,1\right)$ *state vectors;* u being the number of unknowns which in our case is either four or six and the state of the system is its position E_k, N_k and velocity $\dot{E}_k, \dot{N}_k$, or position E_k, N_k, velocity $\dot{E}_k, \dot{N}_k$, and acceleration $\ddot{E}_k, \ddot{N}_k$. The "hat" symbol (^) above the vector $\mathbf{x}$ indicates that it is an estimate of the true (but unknown) state of the system derived from the Kalman filter.

- $\mathbf{T}$ is the (u,u) *Transition Matrix* that models the dynamic relationships between the states at t_{k-1} and t_k.
- $\mathbf{v}_m = \mathbf{Hw}$ is a $(u,1)$ vector of residuals (small unknown corrections) reflecting the fact that the dynamic model is only an approximation of the true (but unknown) model linking the states at t_{k-1} and t_k.
- $\mathbf{H}$ is a coefficient matrix
- $\mathbf{w}$ is the *system driving noise,* which in our case is jerk $\dddot{E}_k, \dddot{N}_k$

A Kalman filter takes an initial estimate of the state vector $\hat{\mathbf{x}}$ and the state cofactor matrix (estimates of precisions) $\mathbf{Q}_x$ at t_{k-1} and predicts $\mathbf{x}'$ and $\mathbf{Q}'_x$ at t_k according to the dynamic model and its associated cofactor matrix. It then updates the predicted quantities using the measurements at t_k and the measurement cofactor matrix, producing new estimates $\hat{\mathbf{x}}$ and $\mathbf{Q}_x$. This process is repeated for successive measurements.

The *primary* measurement model has the general form

$$\mathbf{l}_k + \mathbf{v}_k = \hat{\mathbf{l}}_k \tag{35}$$

- $\mathbf{l}_k$ is the $(n,1)$ vector of measurements
- $\mathbf{v}_k$ is an $(n,1)$ vector of residuals (small unknown corrections to the measurements)
- $\hat{\mathbf{l}}_k$ are estimates of the true (but unknown) measurements.
- n is the number of measurements, which in our case is two, E_{obs} and N_{obs}.

The primary model can be expressed in terms of the state vector as

$$\mathbf{v}_k + \mathbf{B}_k\hat{\mathbf{x}}_k = -\mathbf{l}_k \tag{36}$$

or, using dynamic model (32)

$$\begin{bmatrix} v_E \\ v_N \end{bmatrix}_k + \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E \\ N \\ \dot{E} \\ \dot{N} \\ \ddot{E} \\ \ddot{N} \end{bmatrix}_k = -\begin{bmatrix} E_{obs} \\ N_{obs} \end{bmatrix}_k$$

- $\mathbf{B}_k$ is an (n,u) coefficient matrix and $\mathbf{l}_k$ is a $(u,1)$ vector of observations.

The primary model and the dynamic model have associated *cofactor matrices* $\mathbf{Q}$ and $\mathbf{Q}_m$ that contain estimates of the precision of the measurements and the dynamic model corrections respectively.

$\mathbf{Q}$ is the (n,n) cofactor matrix of the measurements in the primary model

$$\mathbf{Q} = \begin{bmatrix} s_E^2 & s_{EN} \\ s_{EN} & s_N^2 \end{bmatrix} \tag{37}$$

$s_E^2 = s_N^2$ are estimates of the variances of the kinematic GPS coordinates. s_{EN} is an estimate of the covariance between the E and N coordinates. In our case we consider that the E and N coordinates are independent random variables and $s_{EN} = 0$.

$\mathbf{Q}_m$ is the (u,u) cofactor matrix of the of the dynamic model corrections and is obtained by applying the general law of propagation of variances to equation (34) giving

$$\mathbf{Q}_m = \mathbf{H}\mathbf{Q}_w\mathbf{H}^T \tag{38}$$

$\mathbf{Q}_w$ is the (n,n) cofactor matrix of the system driving noise, and in the case of model (32), contains estimates of the variance of the rate of change of acceleration (the jerk) and

$$\mathbf{Q}_m = \begin{bmatrix} \frac{1}{6}(\Delta t)^3 & 0 \\ 0 & \frac{1}{6}(\Delta t)^3 \\ \frac{1}{2}(\Delta t)^2 & 0 \\ 0 & \frac{1}{2}(\Delta t)^2 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix} \begin{bmatrix} s_{\dddot{E}}^2 & 0 \\ 0 & s_{\dddot{N}}^2 \end{bmatrix} \begin{bmatrix} \frac{1}{6}(\Delta t)^3 & 0 & \frac{1}{2}(\Delta t)^2 & 0 & \Delta t & 0 \\ 0 & \frac{1}{6}(\Delta t)^3 & 0 & \frac{1}{2}(\Delta t)^2 & 0 & \Delta t \end{bmatrix} \tag{39}$$

The Kalman filter equations provide the (u,u) cofactor matrix $\mathbf{Q}_x$ containing estimates of the precisions of the elements of the state vector

$$\mathbf{Q}_x = \begin{bmatrix} s_E^2 & s_{EN} & s_{E\dot{E}} & s_{E\dot{N}} & s_{E\ddot{E}} & s_{E\ddot{N}} \\ s_{EN} & s_N^2 & s_{N\dot{E}} & s_{N\dot{N}} & s_{N\ddot{E}} & s_{N\ddot{N}} \\ s_{E\dot{E}} & s_{N\dot{E}} & s_{\dot{E}}^2 & s_{\dot{E}\dot{N}} & s_{\dot{E}\ddot{E}} & s_{\dot{E}\ddot{N}} \\ s_{E\dot{N}} & s_{N\dot{N}} & s_{\dot{E}\dot{N}} & s_{\dot{N}}^2 & s_{\dot{N}\ddot{E}} & s_{\dot{N}\ddot{N}} \\ s_{E\ddot{E}} & s_{N\ddot{E}} & s_{\dot{E}\ddot{E}} & s_{\dot{N}\ddot{E}} & s_{\ddot{E}}^2 & s_{\ddot{E}\ddot{N}} \\ s_{E\ddot{N}} & s_{N\ddot{N}} & s_{\dot{E}\ddot{N}} & s_{\dot{N}\ddot{N}} & s_{\ddot{E}\ddot{N}} & s_{\ddot{N}}^2 \end{bmatrix} \tag{40}$$

All the elements of the primary and the dynamic models have been defined as well as the cofactor matrices associated with both. The primary model at t_{k-1} and t_k, and the dynamic model linking the states at t_{k-1} and t_k give rise to the system of equations

$$\begin{aligned} \mathbf{A}_{k-1}\mathbf{v}_{k-1} + \mathbf{B}_{k-1}\mathbf{x}_{k-1} &= \mathbf{l}_{k-1} \\ \mathbf{A}_k\mathbf{v}_k + \mathbf{B}_k\mathbf{x}_k &= \mathbf{l}_k \\ \mathbf{x}_k &= \mathbf{T}\mathbf{x}_{k-1} + \mathbf{v}_m \end{aligned} \tag{41}$$

Note that in our case $\mathbf{A} = \mathbf{I}$ where $\mathbf{I}$ is the Identity matrix. Enforcing the least squares condition that the sum of the squares of the residuals be a minimum, gives rise to a set of recursive equations (the Kalman Filter) which are applied as follows. [A complete derivation of these equations is given in Deakin (2015)]

With initial estimates of the state vector $\hat{\mathbf{x}}_{k-1}$ and the cofactor matrix $\mathbf{Q}_{x_{k-1}}$ a Kalman Filter has the following five general steps

(1) Project the state forward to give approximate values at t_k

$\mathbf{x}'_k = \mathbf{T}\hat{\mathbf{x}}_{k-1}$

(2) Project the state cofactor matrix forward

$\mathbf{Q}'_{x_k} = \mathbf{T}\mathbf{Q}_{x_{k-1}}\mathbf{T}^T + \mathbf{Q}_m$

(3) Compute the Kalman *Gain matrix*

$$\mathbf{K} = \mathbf{Q}'_{x_k}\mathbf{B}_k^T\left(\mathbf{Q} + \mathbf{B}_k\mathbf{Q}'_{x_k}\mathbf{B}_k^T\right)^{-1}$$

(4) Update the estimate with the measurements at t_k

$$\hat{\mathbf{x}}_k = \mathbf{x}'_k + \mathbf{K}_k\left(\mathbf{l}_k - \mathbf{B}_k\mathbf{x}'_k\right)$$

(5) Update the state cofactor matrix

$$\begin{aligned}\mathbf{Q}_{x_k} &= \left(\mathbf{I} - \mathbf{K}_k\mathbf{B}_k\right)\mathbf{Q}'_{x_k}\left(\mathbf{I} - \mathbf{K}_k\mathbf{B}_k\right)^T + \mathbf{K}_k\mathbf{AQAK}_k^T \\ &= \left(\mathbf{I} - \mathbf{K}_k\mathbf{B}_k\right)\mathbf{Q}'_{x_k}\end{aligned}$$

Go to step (1) and repeat the process for the next measurement epoch.

In the section below, a detailed description of the steps in a Kalman filter [using dynamic model (32)] as implemented in a computer program are set out

Step 1 Set the elements of the transition matrix

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & \Delta t & 0 & \frac{1}{2}\Delta t^2 & 0 \\ 0 & 1 & 0 & \Delta t & 0 & \frac{1}{2}\Delta t^2 \\ 0 & 0 & 1 & 0 & \Delta t & 0 \\ 0 & 0 & 0 & 1 & 0 & \Delta t \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 2 Set the cofactor matrix of the system driving noise

$$\mathbf{Q}_w = \begin{bmatrix} s_{\ddot{E}}^2 & 0 \\ 0 & s_{\ddot{N}}^2 \end{bmatrix}$$

Step 3 Set the coefficient matrix of the system driving noise

$$\mathbf{H} = \begin{bmatrix} \frac{1}{6}\left(\Delta t\right)^3 & 0 \\ 0 & \frac{1}{6}\left(\Delta t\right)^3 \\ \frac{1}{2}\left(\Delta t\right)^2 & 0 \\ 0 & \frac{1}{2}\left(\Delta t\right)^2 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix}$$

Step 4 Compute the cofactor matrix of the dynamic model

$$\mathbf{Q}_m = \mathbf{HQ}_w\mathbf{H}^T$$

Step 5 Set the counter

$$k = 1$$

Step 6 Set the starting estimates of the state vector

$$\hat{\mathbf{x}}_k = \begin{bmatrix} E \\ N \\ \dot{E} \\ \dot{N} \\ \ddot{E} \\ \ddot{N} \end{bmatrix}_k$$

Note that estimates of the starting velocities and accelerations can be computed from kinematic GPS coordinates at epochs 1, 2 and 3

Step 7 Set the starting estimates of the state cofactor matrix

$$\mathbf{Q}_{x_k} = \begin{bmatrix} s_E^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & s_N^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_{\dot{E}}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{\dot{N}}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{\ddot{E}}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{\ddot{N}}^2 \end{bmatrix}_k$$

Note that at this step the estimates of the covariances are all set to zero.

Step 8 Increment the counter

$$k = k + 1$$

Step 9 Compute the predicted state vector

$$\mathbf{x}'_k = \begin{bmatrix} E' \\ N' \\ \dot{E}' \\ \dot{N}' \\ \ddot{E}' \\ \ddot{N}' \end{bmatrix}_k = \begin{bmatrix} 1 & 0 & \Delta t & 0 & \frac{1}{2}\Delta t^2 & 0 \\ 0 & 1 & 0 & \Delta t & 0 & \frac{1}{2}\Delta t^2 \\ 0 & 0 & 1 & 0 & \Delta t & 0 \\ 0 & 0 & 0 & 1 & 0 & \Delta t \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E \\ N \\ \dot{E} \\ \dot{N} \\ \ddot{E} \\ \ddot{N} \end{bmatrix}_{k-1}$$

Step 10 Compute the predicted state cofactor matrix

$$\mathbf{Q}'_{x_k} = \mathbf{T}\mathbf{Q}_{x_{k-1}}\mathbf{T}^T + \mathbf{Q}_m$$

Step 11 Set the elements of the coefficient matrix of the primary model

$$\mathbf{B} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 12 Compute the numeric terms of the primary model

$$\mathbf{f}_k = \begin{bmatrix} E' - E_{obs} \\ N' - N_{obs} \end{bmatrix}_k$$

Step 13 Compute the Kalman *Gain matrix*

$$\mathbf{K}_k = \mathbf{Q}'_{x_k}\mathbf{B}_k^T\left(\mathbf{Q} + \mathbf{B}_k\mathbf{Q}'_{x_k}\mathbf{B}_k^T\right)^{-1}$$

Step 14 Compute corrections to the state vector

$$\Delta\mathbf{x}_k = \mathbf{K}_k\left(\mathbf{l}_k - \mathbf{B}_k\mathbf{x}'_k\right) = \mathbf{K}_k\mathbf{f}_k$$

Step 15 Compute the new estimate of the state vector

$$\mathbf{x}_k = \mathbf{x}'_k + \Delta\mathbf{x}_k$$

Step 16 Compute the cofactor *Update matrix*

$$\mathbf{U}_k = \mathbf{I} - \mathbf{K}_k\mathbf{B}_k$$

Note that I is the identity matrix

Step 17 Compute the new estimate of the state cofactor matrix

$$\mathbf{Q}_{x_k} = \mathbf{U}\mathbf{Q}'_{x_k}$$

Go To Step 8